

RANK-ONE ACTIONS, THEIR (C, F) -MODELS AND CONSTRUCTIONS WITH BOUNDED PARAMETERS

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ABSTRACT. Let G be a discrete countable infinite group. We show that each topological (C, F) -action T of G on a locally compact non-compact Cantor set is a free minimal amenable action admitting a unique up to scaling non-zero invariant Radon measure (answer to a question by Kellerhals, Monod and Rørdam). We find necessary and sufficient conditions under which two such actions are topologically conjugate in terms of the underlying (C, F) -parameters. If G is linearly ordered Abelian then the topological centralizer of T is trivial. If G is monotileable and amenable, denote by \mathcal{A}_G the set of all probability preserving actions of G on the unit interval with Lebesgue measure and endow it with the natural topology. We show that the set of (C, F) -parameters of all (C, F) -actions of G furnished with a suitable topology is a model for \mathcal{A}_G in the sense of Forman, Rudolph and Weiss. If T is a rank-one transformation with bounded sequences of cuts and spacer maps then we found simple necessary and sufficient conditions on the related (C, F) -parameters under which (i) T is rigid, (ii) T is totally ergodic. It is found an alternative proof of Ryzhikov's theorem that if T is totally ergodic and non-rigid rank-one map with bounded parameters then T has MSJ. We also give a simpler and more general version of the criterium (by Gao and Hill) for isomorphism and disjointness of two commensurate non-rigid totally ergodic rank-one maps with bounded parameters. It is shown that the rank-one transformations with bounded parameters and no spacers over the last subtowers is a proper subclass of the rank-one transformations with bounded parameters.

0. INTRODUCTION

The original goal of this work was to find new, short and more explicit proofs of the main results from the recent papers by Gao and Hill [GaHi1]–[GaHi4], [Hi1] and [Hi2] on topological and measure theoretical properties of rank-one transformations. It appeared later that our approach works well to extend those results not only to more general classes of rank-one transformations but also (some of the results) to rank-one actions of more general groups, including non-amenable ones. This is achieved by applying—in our opinion—more natural, intrinsic techniques to the problems under consideration. While Gao and Hill consider the rank-one transformations as shift-maps on invariant subsets of $\{0, 1\}^{\mathbb{Z}}$ and study them via tools of symbolic dynamics, our approach is based on analysis of the standard *cutting-and-stacking* constructing algorithm. Since this classical geometric algorithm loses its clarity beyond the framework of \mathbb{Z}^d -actions, we utilize instead of it the (C, F) -construction (see [dJ], [Da5], [Da6]) which we consider as an *arithmetic version* of the cutting-and-stacking. It is convenient to produce and investigate actions of arbitrary locally compact groups.

In §1 we briefly review the (C, F) -construction and (in the case of \mathbb{Z} -actions) discuss a relation between the (C, F) -notions and the classical concepts related

to the cutting-and-stacking. The class of probability preserving (C, F) -actions of \mathbb{Z} is up to isomorphism the class of \mathbb{Z} -actions of funny rank-one (see [Fe1] for Thouvenot's definition of funny rank one). Every such action is associated with two sequences of finite subsets in \mathbb{Z} : a sequence of *tiling shapes* and a sequence of *tiling centers*. In a similar way, given an arbitrary discrete countable infinite group G and two sequences of finite subsets $(C_n)_{n \geq 1}$ and $(F_n)_{n \geq 0}$ in G satisfying some conditions (see §1.2), we associate a minimal continuous action of G on a perfect totally disconnected Polish space equipped with a canonical σ -finite invariant measure (see §1.3). Under an additional condition (see Lemma 1.4(i)) the space is locally compact and the measure is Radon. This general definition proved to be useful to answer affirmatively the following non-trivial questions in the theory of C^* -algebras and topological group actions (see §1.5):

- does G admit a free minimal amenable (in the topological sense according to Definition 1.6 below) action on a locally compact non-compact Cantor space X [KeMoRø, Question 7.1]¹?
- does G admit a free minimal amenable action on X which leaves invariant a non-zero Radon measure on X [KeMoRø, Question 7.2]?

In §2 we study *topological properties* of continuous (C, F) -actions of G on locally compact Cantor spaces. Our main concern is the topological classification of these actions. Namely, we want to determine when two (C, F) -actions are topologically isomorphic in terms of the underlying sequences $(C_n)_{n \geq 1}$ and $(F_n)_{n \geq 0}$ viewed as the *parameters* of the actions. We completely solve this problem in Theorem 2.3:

Theorem A. *Let $T = (T_g)_{g \in G}$ and $T' = (T'_g)_{g \in G}$ be two (C, F) -actions of G on locally compact Cantor spaces associated with some sequences $(C_n, F_{n-1})_{n \geq 1}$ and $(C'_n, F'_{n-1})_{n \geq 1}$ respectively. Then T and T' are topologically isomorphic if and only if there is an increasing sequence of integers $0 = l_0 < l'_1 < l_1 < l'_2 < l_2 < \dots$ and subsets $A_n \subset F'_{l'_n}$, $B_n \subset F_{l_n}$, such that $A_n B_n = C_{l_{n-1}+1} \cdots C_{l_n}$, $B_n A_{n+1} = C'_{l'_n+1} \cdots C'_{l'_{n+1}}$, $F_{l_n} A_{n+1} \subset F'_{l'_{n+1}}$, $F_{l_n} A_{n+1} \subset F'_{l'_{n+1}}$ and $(F'_{l'_n})^{-1} F'_{l'_n} \cap B_n B_n^{-1} = F_{l_n}^{-1} F_{l_n} \cap A_{n+1} A_{n+1}^{-1} = \{1\}$ for each $n > 0$.*

We show in Theorem 2.6 that this isomorphism criterium is getting especially simple in the case of *linearly ordered* Abelian groups (Definition 2.4) and *commensurate* actions, which means that $F_n = F'_n$ eventually:

Theorem B. *Let T and T' be as above and let (G, G_+) be a linearly ordered discrete countable Abelian group. Suppose that $C_n \cup C'_n \subset G_+$ for all n and $F_n = F'_n$ for all $n > N$ (for some $N > 0$). Then T and T' are topologically isomorphic if and only if $C_n = C'_n$ for all $n > M$ (for some $M > 0$).*

Given a continuous action $(T_g)_{g \in G}$ of G on a topological space X , we call the group of homeomorphisms of X commuting with each T_g , $g \in G$, the *topological centralizer* of T . Denote it by $C_{\text{top}}(T)$. In Corollary 2.7 we characterize the topological centralizer of the (C, F) -actions of linearly ordered groups:

Theorem C. *Let (G, G_+) and T be as in Theorem B. Then $C_{\text{top}}(T) = \{T_g \mid g \in G\}$.*

¹Compare with a recent paper [MaRø] devoted to a partial answer to this question in the case where G is *exact*, i.e. G admits an amenable action on a compact set.

In the case of topological rank-one \mathbb{Z} -actions we provide a satisfactory solution of the *inverse* problem (see Corollary 2.9 for a more general result): when an action is isomorphic to its inverse?

Theorem D. *Let $(G, G_+) = (\mathbb{Z}, \mathbb{Z}_+)$ and let T be as in Theorem C. If $F_n = \{-a_n, \dots, 0, \dots, h_n - 1\}$ for some $a_n, h_n > 0$ and each $n \in \mathbb{N}$ then T_1 and T_{-1} are topologically isomorphic if and only if $C_n = \{\max C_n - c \mid c \in C_n\}$ for all $n > M$ (for some $M > 0$).*

In the particular case where $G = \mathbb{Z}$ and the actions are of topological rank-one (which means that $C_n \subset \mathbb{Z}_+$ and $F_n = \{-a_n, \dots, b_n\}$ for some $a_n, b_n > 0$ and each $n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = +\infty$), Theorems B, C and D are close to the main results from [GaHi2] (and partly from [Hi1]). However there are two points of difference. The topological systems considered by Gao and Hill are defined on compact Cantor spaces while (C, F) -actions from §2 are defined on locally compact non-compact Cantor spaces. This seeming difference is eliminated easily by passing to the one-point compactification. Then the (C, F) -actions extend to these compactifications as *almost minimal* continuous actions on compact Cantor spaces. We recall that a continuous action is called *almost minimal* [Da2] if it has a single fixed point and the other orbits are dense. The second difference is that Gao and Hill consider symbolic models of the so-called *adapted* rank-one transformations. This means that no spacers are added on the top the last (highest) subtower on any step of the inductive cutting-and-stacking-construction. In contrast, we consider (C, F) -models of those rank-one transformations for which spacers are added on infinitely many steps as on the top of the last subtower as under the bottom of the first subtower.

In the rest of the paper (§3–§5) we study *measure theoretical properties* of the (C, F) -actions according to the most general definition (in which (1-4) holds instead of the more restrictive (1-3) considered in §2). Then the (C, F) -actions are defined on Polish but not necessarily locally compact spaces and they have a canonical ergodic invariant σ -finite measure. Thus we regard the (C, F) -actions in §3–§5 as standard *measure preserving* dynamical systems and study them by modulo measure theoretical isomorphism. In particular, in the case where $G = \mathbb{Z}$, the adapted rank-one transformations considered by Gao and Hill in [GaHi1], [GaHi3], [GaHi4] and [Hi2] are all in this class of (C, F) -systems.

In §3, G is a *monotileable* amenable group [We]. Let \mathcal{F} be a Følner sequence of finite sets that tile G . We denote by \mathcal{A}_G the set of all Lebesgue measure preserving actions of G on the unit interval $[0, 1)$. Endow this set with the natural (Polish) weak topology. We generalize the concept of a *model* for \mathbb{Z} -actions in the sense of Forman-Rudolph-Weiss [Fo] to the case of G -actions (Definition 3.5).

Definition E. A *model* for \mathcal{A}_G is a pair (W, π) , where W is a Polish space and $\pi : W \rightarrow \mathcal{A}_G$ is a continuous map such that for a comeager set $\mathcal{M} \subset \mathcal{A}_G$ and each $A \in \mathcal{M}$, the set $\{w \in W \mid \pi(w) \text{ is isomorphic to } A\}$ is dense in W .

Denote by $\mathfrak{R}_1^{\text{fin}}$ the set of all possible (C, F) -parameters, i.e. sequences $(C_n)_{n \geq 1}$ and $(F_n)_{n \geq 1}$, satisfying the conditions for (C, F) -actions and for which the corresponding canonical measure is finite. Thus $\mathfrak{R}_1^{\text{fin}}$ is a subset of $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$, where \mathfrak{F} is the countable set of all finite subsets in G . Then we introduce a certain Polish topology on $\mathfrak{R}_1^{\text{fin}}$ which is stronger than the product topology inherited from $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$. We also construct a continuous map $\Psi : \mathfrak{R}_1^{\text{fin}} \ni \mathcal{S} \mapsto \Psi(\mathcal{S}) \in \mathcal{A}_G$ such

that $\Psi(\mathcal{S})$ is isomorphic to the (C, F) -action associated with \mathcal{S} for each $\mathcal{S} \in \mathfrak{R}_1^{\text{fin}}$. The following is the main result of §3 (see Proposition 3.6 and Corollary 3.7).

Theorem F. *The subset of G -actions which are of rank one along \mathcal{F} is a dense G_δ in \mathcal{A}_G . The pair $(\mathfrak{R}_1^{\text{fin}}, \Psi)$ is a model for \mathcal{A}_G .*

In the particular case where $G = \mathbb{Z}$ and $\mathcal{F} = \{[0, \dots, n] \mid n \in \mathbb{N}\}$, the second claim of Theorem F is an alternative version of the main result from [GaHi1].

In §4, we consider rank-one \mathbb{Z} -actions with *bounded parameters*. This means that the number of cuts and the total number of spacers added on each step of the inductive cutting-and-stacking construction are both bounded. Equivalently, in the language of the (C, F) -construction, the parameters $(C_n)_{n \geq 1}$ and $(F_n)_{n \geq 0}$ of a (C, F) -action of \mathbb{Z} are bounded if the sequence $(\#C_n)_{n \geq 1}$ is bounded and there is a finite subset $K \subset \mathbb{Z}$ such that $K + F_n + C_{n+1} \supset F_{n+1}$ and $F_n \in \mathcal{F} := \{[0, \dots, n] \mid n \in \mathbb{N}\}$ for each $n \geq 0$. We note that each rank-one \mathbb{Z} -action with bounded parameters is finite measure preserving. The interest to such systems grew up after Bourgain's work [Bo] where it was shown that they satisfy the Möbius orthogonality property². We now state the main results of §4 (see Theorems 4.4, 4.7 and Corollary 4.8).

Theorem G. *Let T be a (C, F) -action of \mathbb{Z} associated with bounded parameters $(C_n, F_{n-1})_{n \geq 1}$ and let $F_n \in \mathcal{F}$ for each $n > 0$ ³. Then T is rigid if and only if for each $N > 0$, there are integers n, m such that $m > n + N > n > N$ and the set $C_n + \dots + C_m$ is an arithmetic sequence.*

Theorem H. *Let $T = (T_g)_{g \in \mathbb{Z}}$ be a (C, F) -action of \mathbb{Z} associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ and let $F_n \in \mathcal{F}$ for each $n > 0$ ⁴.*

- (i) *If T_d is ergodic then for each divisor p of d , there are infinitely many $n > 0$ such that some $c \in C_n$ is not divisible by p .*
- (ii) *If the sequence $(\#C_n)_{n=1}^\infty$ is bounded and for each divisor p of a positive integer d , there are infinitely many n such that p does not divide some $c \in C_n$ then T_d is ergodic.*
- (iii) *If the sequence $(\#C_n)_{n=1}^\infty$ is bounded then T is totally ergodic if and only if for each $d > 1$, there are infinitely many $n > 0$ such that some element c of C_n is not divisible by d .*

The above two theorems generalize the main results from [GaHi3] where only adapted (and finite measure preserving) rank-one transformations were under consideration.

In [Ry], Ryzhikov stated a theorem that the totally ergodic and non-rigid rank-one transformations with bounded parameters have the property of *minimal self-joinings (MSJ)*. We refer to [dJRu] and [Ru] for the definition of MSJ. The theorem generalizes the well known result from [dJRaSw] that the Chacon transformation with 3 cuts has MSJ. Ryzhikov provided a sketch of a proof that is based on the limit properties of joinings and the weak limits of powers. In [GaHi4], Gao and Hill present a different proof of this result via tools of symbolic dynamics. However they do only a particular case of Ryzhikov's theorem because they consider only adapted rank-one transformations (see Theorem L and the remark just below it).

²See also subsequent works [EALedR], [Ry].

³The condition $F_n \in \mathcal{F}$ for each n means that T is of rank one.

⁴We do not assume here that T preserves finite measure. It can be infinite (σ -finite) measure preserving as well.

In §5, we provide a detailed proof of the *full version* of Ryzhikov's theorem in the framework of (C, F) -construction (see Theorem 5.3 and Corollary 5.4):

Theorem I. *Let T be a rank-one \mathbb{Z} -action with bounded parameters. Suppose that T is not rigid and that T is totally ergodic. Then T has MSJ. Hence T_n and T_m are disjoint⁵ for all $n \neq m \in \mathbb{N}$.*

Our proof is based on standard analysis of generic points for the self-joinings of the rank-one maps under question (neither weak limits of powers nor limit properties of joinings appear in our proof). As a byproduct, we find a criterium for isomorphism and disjointness (in Furstenberg sense [Fu]) for commensurate non-rigid rank-one transformations with bounded parameters.

Theorem J. *Let T and T' be two (C, F) -action of \mathbb{Z} associated with bounded parameters $(C_n, F_{n-1})_{n \geq 1}$ and $(C'_n, F'_{n-1})_{n \geq 1}$ and let $F_n = F'_n \in \mathcal{F}$ eventually. Let T not be rigid.*

- (i) *Then T and S are isomorphic if and only if $C_n = C'_n$ eventually.*
- (ii) *If $C_n \neq C'_n$ for infinitely many n and for each $n > 0$, either T_n or S_n is ergodic then T and S are disjoint.*

As an application we obtain a satisfactory solution of the *measure theoretical inverse* problem within the class of non-rigid rank-one transformations with bounded parameters (cf. with Theorem D above).

Theorem K. *Let T be a (C, F) -action of \mathbb{Z} associated with bounded parameters $(C_n, F_{n-1})_{n \geq 1}$ and let $F_n \in \mathcal{F}$ eventually. If T is not rigid then T_1 and T_{-1} are isomorphic if and only if $C_n = \{\max C_n - c \mid c \in C_n\}$ eventually. If, moreover, T is totally ergodic and T_1 is not isomorphic to T_{-1} then T_1 and T_{-1} are disjoint.*

We note that Theorems J and K extend the main results from [GaHi4] and [Hi2], where only adapted rank-one transformations were under consideration. In this connection a natural question arises: is the class of adapted rank-one transformations with bounded parameters (considered up to measure theoretical isomorphism) is really less than the class of all rank-one transformations with bounded parameters? If we drop the boundedness restriction then the two classes coincide (see Lemma 1.10). The affirmative answer follows from the next theorem.

Theorem L. *Let T be an adapted rank-one action of \mathbb{Z} with bounded parameters⁶. Then there is a sequence $n_m \rightarrow +\infty$ and a polynomial $P(Z) = \nu_0 + \nu_1 Z + \dots + \nu_K Z^K$ with non-negative real coefficients ν_i such that $\sum_{0 \leq i \leq K} \nu_i = 1$ and $T_{-n_m} \rightarrow P(T_1)$ as $m \rightarrow \infty$ in the weak operator topology⁷. It follows that T is not lightly mixing.*

Since the Chacon map with 2 cuts is lightly mixing [FrKi], it follows that it is not isomorphic to any adapted rank-one transformation with bounded parameters.

The following quadchotomy theorem for rank-one transformations with bounded parameters refines (with a different proof) the trichotomy theorem [EALeRu, Theorem 3].

⁵A stronger result that they are spectrally disjoint was proved in [EALeRu].

⁶In fact, it suffices to claim that only the sequence of spacers is bounded.

⁷We identify T_n here with the unitary Koopman operators in $L^2(X, \mu)$ generated by it, $n \in \mathbb{Z}$.

Theorem M. *Let T be a rank-one \mathbb{Z} -action with bounded parameters. Let K denote the upper bound for the number of spacers put on a subcolumn over all subcolumns and all steps of the inductive cutting-and-stacking construction. Then one of the four possibilities takes place:*

- (i) *T has MSJ (in particular, T is weakly mixing and $C(T) = \{T_n \mid n \in \mathbb{Z}\}$),*
- (ii) *T is non-rigid, the group $\Lambda_T \subset \mathbb{T}$ of eigenvalues of T is nontrivial but finite and the order of each $\lambda \in \Lambda_T$ does not exceed K . For each ergodic 2-fold self-joining of T which is neither a graph-joining nor $\mu \times \mu$, there is $\lambda \in \Lambda_T \setminus \{1\}$ and $n > 0$ such that $\lambda^n = 1$ and $\frac{1}{n} \sum_{i=0}^{n-1} \rho \circ (I \times T_i) = \mu \times \mu$.*
- (iii) *T is rigid, the group $\Lambda_T \subset \mathbb{T}$ of eigenvalues of T is finite and the order of each $\lambda \in \Lambda_T$ does not exceed K .*
- (iv) *T is an odometer of bounded type⁸.*

After this paper had been submitted of this paper I learned about a recent work [AdFePe] by Adams, Ferenczi and Petersen. The main result of this work is that *every* probability preserving rank-one map defined by the cutting-and-stacking construction process admits a *symbolic presentation* as a uniquely ergodic binary subshift. Thus the *constructive symbolic definition* of rank one given in the famous survey [Fe2] (and in a sense conflicting with the usual definition of odometers) is equivalent to the standard cutting-and-stacking one. Then I wrote Appendix, where I use the (C, F) -machinery⁹ to give a short alternative proof of this result. It follows from the next theorem:

Theorem N. *Each finite measure preserving rank-one transformation is isomorphic to a rank-one transformation $(X, \mathfrak{B}, \mu, T)$ that is essentially 0-expansive, i.e. the smallest T -invariant sub- σ -algebra containing the initial level of the cutting-and-stacking inductive construction of T is $\mathfrak{B} \pmod{\mu}$.*

Theorem N follows from the fact that each finite measure preserving rank-one transformation is isomorphic to a rank-one transformation defined by the cutting-and-stacking inductive construction in such a way that on each step of this construction, the number of spacers put over the last subtower is strictly greater than the number of spacers put over each other subtower. This fact is, in turn, a slight refinement of a particular case of [Da6, Theorem 2.8].

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1. (C, F) -CONSTRUCTION OF RANK-ONE ACTIONS

1.1. Actions of rank one. Let G denote a discrete countable infinite group. Fix an infinite sequence $\mathcal{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ of finite subsets in G . Let $T = (T_g)_{g \in G}$ be a measure preserving action of G on a standard σ -finite measure space (X, \mathfrak{B}, μ) .

Definition 1.1. If there exist a sequence $(B_n)_{n \geq 0}$ of subsets of finite measure in X and an increasing sequence $(l_n)_{n \geq 0}$ of non-negative integers such that

- (i) for each $n \geq 0$, the subsets $T_g B_n$, $g \in \mathcal{F}_{l_n}$, are pairwise disjoint and

⁸For the definition of odometers of bounded type see several lines above Proposition 4.5.

⁹Adams, Ferenczi and Petersen use the Bratteli-Vershik models of rank-one maps to prove the main result of [AdFePe].

(ii) for each subset $B \in \mathfrak{B}$ with $\mu(B) < \infty$,

$$\lim_{n \rightarrow \infty} \inf_{F \subset \mathcal{F}_{l_n}} \mu \left(B \triangle \bigsqcup_{g \in F} T_g B_n \right) = 0$$

then we say that T is of *rank one*¹⁰ along \mathcal{F} .

If $G = \mathbb{Z}$ and $\mathcal{F} = (\{0, 1, \dots, n\})_{n \geq 0}$ then we obtain the standard definition of *rank-one transformations* (or \mathbb{Z} -actions). If $G = \mathbb{Z}$ but \mathcal{F} is the set of all finite subsets of G then we obtain the definition of transformations of *funny rank one*. If $G = \mathbb{Z}^d$ for $d > 1$ and $\mathcal{F} = (\{0, 1, \dots, n\}^d)_{n \geq 0}$ then we obtain the definition of \mathbb{Z}^d -actions of *rank one along cubes*.

Remark 1.2. We note that if $T = (T_g)_{g \in G}$ is of rank one along \mathcal{F} then we can assume without loss of generality that the following property holds in addition to (i) and (ii):

(iii) for each $n \geq 0$, there is a subset $C_n \subset F_{l_{n+1}}$ such that $B_n = \bigsqcup_{g \in C_n} T_g B_{n+1}$.

We leave the proof of this standard claim as an exercise to the reader.

1.2. (C, F) -spaces, canonical measures and tail equivalence relations. For a detailed exposition of the (C, F) -concepts we refer to [Da1], [Da5] and [Da6]. Let $(F_n)_{n \geq 0}$ and $(C_n)_{n \geq 1}$ be two sequences of finite subsets in G such that for each $n > 0$,

- (I) $1 \in F_0 \cap C_n$, $\#C_n > 1$,
- (II) $F_n C_{n+1} \subset F_{n+1}$,
- (III) $F_n c \cap F_n c' = \emptyset$ if $c, c' \in C_{n+1}$ and $c \neq c'$.

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \dots$ and endow this set with the infinite product topology. Then X_n is a compact Cantor (i.e. totally disconnected perfect metric) space. The mapping

$$X_n \ni (f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n c_{n+1}, c_{n+2}, \dots) \in X_{n+1}$$

is a topological embedding of X_n into X_{n+1} . Therefore an inductive limit X of the sequence $(X_n)_{n \geq 0}$ furnished with these embeddings is a well defined locally compact Cantor space. We call it the (C, F) -space associated with the sequence $(C_n, F_{n-1})_{n \geq 1}$. It is easy to see that the (C, F) -space is compact if and only if there is $N > 0$ with $F_{n+1} = F_n C_{n+1}$ for all $n > N$. Given $n \geq 0$ and a subset $A \subset F_n$, we let

$$[A]_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n \in A\}$$

and call this set an n -cylinder in X . It is open and compact in X . The collection of all cylinders coincides with the family of all compact open subsets in X . It is easy to see that

$$(1-1) \quad \begin{aligned} [A]_n \cap [B]_n &= [A \cap B]_n, & [A]_n \cup [B]_n &= [A \cup B]_n \quad \text{and} \\ [A]_n &= [A C_{n+1}]_{n+1} \end{aligned}$$

for all $A, B \subset F_n$ and $n \geq 0$. For brevity, we will write $[f]_n$ for $[\{f\}]_n$, $f \in F_n$.

¹⁰Sometimes T is called of *funny rank one*.

Let \mathcal{R} denote the *tail equivalence relation* on X . This means that for each $n \geq 0$, the restriction of \mathcal{R} to X_n is the tail equivalence relation on X_n , i.e. two points (f_n, c_{n+1}, \dots) and (f'_n, c'_{n+1}, \dots) from X_n are equivalent if there is $N > 0$ such that $c_m = c'_m$ for all $m > N$. We note that \mathcal{R} is *minimal*, i.e. the \mathcal{R} -class of every point is dense in X and *uniquely ergodic*, i.e. there exists a unique up to scaling non-zero σ -finite \mathcal{R} -invariant¹¹ Radon¹² measure μ on X . Moreover, μ is strictly positive on every non-empty open subset. We note that the \mathcal{R} -invariance of μ is equivalent to the following property:

$$\mu([f]_n) = \mu([f']_n) \quad \text{for all } f, f' \in F_n, n \geq 0.$$

Using this property and (1-1) we can compute that

$$\mu([A]_n) = \frac{\mu([1]_0) \#A}{\#C_1 \cdots \#C_n} \quad \text{for each subset } A \subset F_n, n > 0.$$

In what follows we normalize μ by the condition $\mu([1]_0) = 1$. We see that the restriction of μ to X_0 is the product of the “counting” measure on F_0 with the infinite product of equidistributions on C_n , $n \in \mathbb{N}$. We call μ the *canonical measure associated with* $(C_n, F_{n-1})_{n \geq 1}$. It is finite if and only if¹³

$$(1-2) \quad \lim_{n \rightarrow \infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} < \infty.$$

It is easy to see that μ on is \mathcal{R} -ergodic, i.e. each Borel \mathcal{R} -saturated subset of X is either μ -null or μ -conull.

1.3. (C, F) -actions. We now define an action of G on X (or, more rigorously, on a subset of X). Given $g \in G$, let

$$X_n^g := \{(f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \mid g f_n \in F_n\}.$$

Then X_n^g is a compact open subset of X_n and $X_n^g \subset X_{n+1}^g$. Hence the union $X^g := \bigcup_{n \geq 0} X_n^g$ is a well defined open subset of X . Let $X^G := \bigcap_{g \in G} X^g$. Then X^G is a G_δ -subset of X . Hence X^G is Polish in the induced topology. Given $x \in X^G$ and $g \in G$, there is $n > 0$ such that $x = (f_n, c_{n+1}, \dots) \in X_n$ and $g f_n \in F_n$. We now let

$$T_g x := (g f_n, c_{n+1}, \dots) \in X_n \subset X.$$

It is standard to verify that

- (i) $T_g x \in X^G$,
- (ii) the map $T_g : X^G \ni x \mapsto T_g x \in X^G$ is a homeomorphism of X^G and
- (iii) $T_g T_{g'} = T_{gg'}$ for all $g, g' \in G$.

Thus $T := (T_g)_{g \in G}$ is a continuous action of G on X^G .

¹¹ μ is called \mathcal{R} -invariant if μ is invariant under each Borel transformation whose graph is contained in \mathcal{R} .

¹²i.e. it is finite on every compact subset.

¹³In view of (I)–(III), the sequence $(\frac{\#F_n}{\#C_1 \cdots \#C_n})_{n=1}^\infty$ is non-decreasing and bounded by $\#F_0$ from below.

Definition 1.3. We call T the (C, F) -action of G associated with the sequence $(C_n, F_{n-1})_{n \geq 0}$.

Each (C, F) -action is free (except for the trivial case where $X^G = \emptyset$). It is obvious that X^G is \mathcal{R} -invariant and the T -orbit equivalence relation is the restriction of \mathcal{R} to X^G . It follows that T preserves μ . Since X^G is \mathcal{R} -saturated and μ is \mathcal{R} -ergodic, we have either $\mu(X^G) = 0$ or $\mu(X \setminus X^G) = 0$. Each of the two cases can occur.

Lemma 1.4 [Da6, Theorem 1.5].

(i) $X^G = X$ if and only if for each $g \in G$ and $n > 0$, there is $m > n$ such that

$$(1-3) \quad gF_n C_{n+1} C_{n+2} \cdots C_m \subset F_m.$$

(ii) $\mu(X \setminus X^G) = 0$ if and only if for each $g \in G$ and $n > 0$,

$$(1-4) \quad \lim_{m \rightarrow \infty} \frac{\#((gF_n C_{n+1} C_{n+2} \cdots C_m) \cap F_m)}{\#F_n \#C_{n+1} \cdots \#C_m} = 1.$$

(iii) If $\mu(X) < \infty$ then $\mu(X \setminus X^G) = 0$ if and only if $(F_n)_{n \geq 0}$ is a Følner sequence in G and hence G is amenable.

We also note that T is of rank one along $(F_n)_{n \geq 0}$. The converse assertion holds also.

Lemma 1.5 ([Da6, Theorems 1.6, 1.8]). If T is a σ -finite measure preserving G -action of rank one along \mathcal{F} then T is (measure theoretically) isomorphic to a (C, F) -action of G on the (C, F) -space equipped with the canonical measure. Moreover, without loss of generality we may assume that the corresponding (C, F) -sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfy (1-3) and $(F_n)_{n \geq 0}$ is a subsequence of \mathcal{F} .

Remark 1.6. We note that if $X^G = X$ but X is not compact then T extends to the one-point compactification $X^* = X \sqcup \{\infty\}$ of X by setting $T_g \infty = \infty$ for all $g \in G$. We thus obtain a continuous action of G on the compact Cantor space X^* . This action is *almost minimal*, i.e. there is one fixed point and the orbit of any other point is dense. This concept was introduced in [Da2] in the case $G = \mathbb{Z}$.

1.4. Telescoping. We now introduce an important concept of telescoping for the (C, F) -sequences. Given a non-decreasing sequence $(k_n)_{n \geq 0}$ of non-negative integers, we let $\tilde{F}_n := F_{k_n}$ and $\tilde{C}_n := C_{k_{n-1}+1} C_{k_{n-1}+2} \cdots C_{k_n}$. We call the sequence $(\tilde{C}_n, \tilde{F}_{n-1})_{n > 0}$ the $(k_n)_{n \geq 0}$ -telescoping of $(C_n, F_{n-1})_{n > 0}$. It is easy to see that if $(C_n, F_{n-1})_{n > 0}$ satisfies (I)–(III) and (1-3) (or (1-4)) then $(\tilde{C}_n, \tilde{F}_{n-1})_{n > 0}$ satisfies the same conditions. Denote by \tilde{T} the (C, F) -action associated with $(\tilde{C}_n, \tilde{F}_{n-1})_{n > 0}$. Then \tilde{T} is canonically isomorphic to T . Indeed, let X and \tilde{X} denote the corresponding (C, F) -spaces and

$$X = \bigcup_{n \geq 0} X_n = \bigcup_{n \geq 0} X_{k_n} \quad \text{and} \quad \tilde{X} = \bigcup_{n \geq 0} \tilde{X}_n,$$

where $X_n = F_n \times C_{n+1} \times \cdots$ and $\tilde{X}_n = \tilde{F}_n \times \tilde{C}_{n+1} \times \cdots$. Then the mappings

$$X_{k_n} \ni (f_{k_n}, c_{k_n+1}, \dots) \mapsto (f_{k_n}, c_{k_n+1} \cdots c_{k_{n+1}}, c_{k_{n+1}+1} \cdots c_{k_{n+2}}, \dots) \in \tilde{X}_n,$$

$n \geq 0$, define a homeomorphism of X onto \tilde{X} . It is easy to see that this homeomorphism intertwines T with \tilde{T} and the tail equivalence relation on X with the tail equivalence relation on \tilde{X} .

If $\sup_{n \geq 0} (k_{n+1} - k_n) < \infty$ we call the $(k_n)_{n \geq 0}$ -telescoping *bounded*.

1.5. An application to topological group actions. We now show how to use the (C, F) -construction (plus Proposition 1.8 below) to answer some questions in the theory of topological group actions stated in §0.

Definition 1.7 [An, Proposition 2.2]. A continuous action $T = (T_g)_{g \in G}$ of G on a locally compact second countable space X is called *amenable* if there exists a sequence $(g_i)_{i=1}^\infty$ of nonnegative continuous functions on $X \times G$ such that

- (a) for every $i \in \mathbb{N}$ and $x \in X$, $\int_G g_i(x, t) dt = 1$;
- (b) $\lim_{i \rightarrow \infty} \int_G |g_i(T_s x, st) - g_i(x, t)| dt = 0$ uniformly on compact subsets of $X \times G$.

Proposition 1.8. *Let T be a (C, F) -action of G associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(III) and (1-3). Then T is amenable.* ■

Proof. For each $i > 0$, we define $g_i : X \times G \rightarrow \mathbb{R}_+$ by setting

$$g_i(x, h) := \begin{cases} \frac{1}{\#F_i} 1_{F_i}(h^{-1} f_i) & \text{if } x = (f_i, c_{i+1}, \dots) \in X_i := F_i \times C_{i+1} \times \dots \\ \frac{1}{\#F_i} 1_{F_i}(h^{-1}) & \text{if } x \notin X_i. \end{cases}$$

Then g_i is continuous and $\int_G g_i(x, h) dh = 1$ for each $x \in X$ and $i > 0$. Now fix $n > 0$ and $s \in G$. It follows from (1-3) that there is $i > n$ such that $F_n \cup sF_n C_{n+1} \cdots C_i \subset F_i$ and hence $X_n \cup T_s X_n \subset X_i$. Therefore for each $x = (f_n, c_{n+1}, \dots) \in X_n$, we have

$$T_s x = T_s(f_i, c_{i+1}, \dots) = (s f_i, c_{i+1}, \dots),$$

where $f_i = f_n c_{n+1} \cdots c_i$. This yields $g_i(T_s x, sh) = g_i(x, h)$ and

$$\lim_{i \rightarrow \infty} \max_{x \in X_n} \int_G |g_i(x, h) - g_i(T_s x, sh)| dh = 0.$$

Hence T is amenable. \square

Thus the action T is a free minimal amenable action of G on a locally compact Cantor space and T leaves invariant a unique (up to scaling) non-trivial Radon measure.

1.6. (C, F) -concepts and the classical “cutting-and-stacking” nomenclature in case of \mathbb{Z} -actions. Suppose that $G = \mathbb{Z}$. We recall the classical cutting-and-stacking construction of rank-one transformations (see, e.g. [Ru]). A *tower* A is an ordered finite collection of pairwise disjoint intervals (called the *levels* of A) in \mathbb{R} , each of the same Lebesgue measure. We think of the levels in a tower as being stacked on top of each other, so that the $(j + 1)$ -st level is directly above the j -th level. Every tower $A = (I_j)_j$ is associated with a natural tower map T_A sending each point in I_j to the point directly above it in I_{j+1} . A rank-one cutting-and-stacking construction of a measure preserving transformation T consists of a sequence of towers $(A_n)_{n \geq 0}$ such that A_0 is a single interval $[0, 1)$, each tower A_{n+1} is obtained from A_n by cutting A_n into $r_n \geq 2$ subtowers of equal width, adding some number $\sigma_n(k)$ of new levels (called *spacers*) above the k -th subcolumn, $k = 0, \dots, r_n - 1$, and stacking every subtower under the subtower to its right. We note the spacers are intervals drawn from \mathbb{R} that are disjoint from the levels of A_n and the other spacers added to it. They are of the same length as the levels of the subcolumns of A_n . It

is easy to see that $T_{A_{n+1}} \upharpoonright A_n = T_{A_n}$ for each n . We now set $X := \bigcup_{n \geq 0} \bigsqcup_{I \in A_n} I$, endow X with the Lebesgue measure and define T to be the pointwise limit of T_{A_n} as $n \rightarrow \infty$. Then T is a measure preserving invertible transformation of X . We note that T is completely defined by the sequence of integers $(r_n)_{n \geq 1}$ and the sequence $(\sigma_n)_{n \geq 1}$ of maps $\sigma_n : \{1, \dots, r_n\} \rightarrow \mathbb{Z}_+$. We denote this fact by writing $T \sim (r_n, \sigma_n)_{n=1}^\infty$. For example, if we let $r_n = 2$, $\sigma_n(1) = 0$ and $\sigma_n(2) = 1$ for each $n \in \mathbb{N}$ then the rank-one transformation $T \sim (r_n, \sigma_n)_{n=1}^\infty$ is the *Chacon 2-cuts map*. If $r_n = 3$, $\sigma_n(1) = \sigma_n(3) = 0$ and $\sigma_n(2) = 1$ for each $n \in \mathbb{N}$ then the rank-one transformation $T \sim (r_n, \sigma_n)_{n=1}^\infty$ is the *Chacon 3-cuts map*.

We now show how to obtain T via the (C, F) -construction. For that we set $h_0 := 0$, $h_{n+1} := h_n r_n + \sum_{i=1}^{r_n} \sigma_n(i)$, $F_n := \{0, 1, \dots, h_n - 1\}$, $s_n(0) := 0$, $s_n(i) := \sum_{j \leq i} \sigma_n(j)$ if $i = 1, \dots, r_n - 1$ and $C_{n+1} := \{i h_n + s_n(i) \mid i = 0, \dots, r_n - 1\}$. In the sequel we will call s_n the *integral* of σ_n . Then the sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfies (I)–(III). Moreover, it satisfies (1-4) because $(F_n)_{n \geq 0}$ is a Følner sequence in \mathbb{Z} . Hence the associated (C, F) -action of \mathbb{Z} is well defined. It is standard to see that this action is isomorphic to $(T^n)_{n \in \mathbb{Z}}$ by an isomorphism that identifies (uniquely, in accordance with the orders) the levels of A_n with the cylinders $\{[f]_n \mid f \in F_n\}$ for each $n > 0$ (the order on F_n is inherited from the standard linear order on \mathbb{Z}).

Conversely, let $\mathcal{F} := \{\{0, \dots, n\} \mid n \in \mathbb{N}\}$ and let $T = (T_g)_{g \in \mathbb{Z}}$ be a (C, F) -action associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ such that $F_n \in \mathcal{F}$ for every n . Then $T_1 \sim (r_n, \sigma_n)_{n=0}^\infty$, where $r_n := \#C_{n+1}$, $h_n := \#F_n$,

$$(1-5) \quad \sigma_n(i) := \begin{cases} s_n(i) - s_n(i-1) & \text{if } 1 \leq i < r_n, \\ h_{n+1} - r_n h_n - s_n(r_n - 1) & \text{if } i = r_n, \end{cases}$$

and s_n is the unique map $s_n : \{0, \dots, r_n - 1\} \rightarrow \mathbb{Z}_+$ such that

$$(1-6) \quad C_{n+1} = \{i h_n + s_n(i) \mid i = 0, \dots, r_n - 1\}.$$

We also note that for each $n \geq 0$, the pair (C_{n+1}, F_{n+1}) uniquely defines (and, conversely, is uniquely defined by) the pair (r_n, σ_n) via (1-5) and (1-6). We will denote this correspondence by $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$.

Definition 1.9. We say that a rank-one transformation $T \sim (r_n, \sigma_n)_{n=1}^\infty$ is *adapted* if $\sigma(r_n) = 0$ for each $n \geq 1$.

The following claim is a folklore (at least, in the case of finite invariant measure). Unfortunately, we were unable to find a proof of this simple claim in the literature. Therefore we provide below an idea of its proof.

Lemma 1.10. *Each rank-one transformation T is (measure theoretically) isomorphic to an adapted one.*

Idea of the proof. Let $T \sim (r_n, \sigma_n)_{n=1}^\infty$ for some integers $r_n > 1$ and maps $\sigma_n : \{1, \dots, r_n\} \rightarrow \mathbb{Z}_+$, $n \in \mathbb{N}$. We now define a new sequence of maps $\tilde{\sigma}_n : \{1, \dots, r_n\} \rightarrow \mathbb{Z}_+$, $n \in \mathbb{N}$ by setting

$$\tilde{\sigma}_1(i) := \begin{cases} \sigma_1(i) & \text{if } i < r_1, \\ 0 & \text{if } i = r_1 \end{cases}$$

and

$$\tilde{\sigma}_n(i) := \begin{cases} \sigma_1(i) + \dots + \sigma(r_{n-1}) + \sigma_n(i) & \text{if } i < r_n, \\ 0 & \text{if } i = r_n \end{cases}$$

for $n > 1$. Let \tilde{T} stand for the rank-one transformation defined by the sequence $(r_n, \tilde{\sigma}_n)_{n=0}^\infty$. Of course, \tilde{T} is adapted.

As was shown above, we can assume without loss of generality that $(T^n)_{n \in \mathbb{Z}}$ is the (C, F) -action associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ such that $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$ for each n and $(\tilde{T}^n)_{n \in \mathbb{Z}}$ is the (C, F) -action associated with a sequence $(\tilde{C}_n, \tilde{F}_{n-1})_{n \geq 1}$ such that $(\tilde{C}_{n+1}, \tilde{F}_{n+1}) \sim (r_n, \tilde{\sigma}_n)$ for each n . It is a routine to verify that $C_n = \tilde{C}_n$ for each $n > 0$. Then it follows from Lemma 5.8 below that T and \tilde{T} are isomorphic. \square

2. TOPOLOGICAL CLASSIFICATION OF (C, F) -ACTIONS

2.1. Topological isomorphism for general (C, F) -actions. In this subsection we investigate when two (C, F) -actions defined on locally compact Cantor spaces are topologically isomorphic. We first prove an auxiliary lemma.

Lemma 2.1. *Let $T = (T_g)_{g \in G}$ and $T' = (T'_g)_{g \in G}$ be two minimal free G -actions on locally compact Cantor spaces X and X' respectively. Let A be a compact open subset in X and let A' be a compact open subset in X' . If there is a homeomorphism $\theta : A \rightarrow A'$ such that $\theta T_g x = T'_g \theta x$ for each $x \in A$ and $g \in G$ such that $T_g x \in A$ then T and S are conjugate. The homeomorphism from X to X' intertwining T with T' and extending θ is unique.*

Proof. Given $x \in X$, there is $g \in G$ such that $T_g x \in A$. This follows from the fact that T is minimal. We now set $\tilde{\theta} x := T'_{g^{-1}} \theta T_g x \in Y$. We first check that $\tilde{\theta}$ is well defined, i.e. if $T_h x \in A$ for some $h \in G$ then $T'_{h^{-1}} \theta T_h x = T'_{g^{-1}} \theta T_g x$. Indeed, we have $T_{hg^{-1}} T_g x \in A$ and hence $\theta T_h x = \theta T_{hg^{-1}} T_g x = T'_{hg^{-1}} \theta T_g x$, as desired.

Next, we claim that $\tilde{\theta}$ is one-to-one. Indeed, if $T'_{g^{-1}} \theta T_g x = T'_{h^{-1}} \theta T_h y$ for some $x, y \in A$ and $g, h \in G$ with $T_g x \in A$ and $T_h y \in A$ then $T'_{gh^{-1}} \theta T_{hg^{-1}} T_g y = \theta T_g x$. It follows that $\theta T_g y = \theta T_g x$ by the “equivariant” property of θ . Since θ is one-to-one and T_g is one-to-one, we obtain that $y = x$.

We now show that $\tilde{\theta}$ is onto. Take $y \in X'$. Since T' is minimal, there is $g \in G$ such that $T'_g y \in A'$. We let $x := T_{g^{-1}} \theta^{-1} T'_g y$. Then $T_g x \in A$ and hence $\tilde{\theta} x := T'_{g^{-1}} \theta T_g x = y$.

It follows easily from the definition of $\tilde{\theta}$ that $\tilde{\theta}$ is continuous at every point of X . Since X is sigma-compact and θ is a bijection of X onto X' , the mapping θ^{-1} is continuous everywhere on X' .

It is straightforward to verify that $\tilde{\theta} T_g = T'_g \tilde{\theta}$ for each $g \in G$.

The final claim of the lemma is obvious. \square

Corollary 2.2. *Let T and T' be two (C, F) -actions of G associated with sequences $(C_n, F_{n-1})_{n \geq 1}$ and $(C'_n, F'_{n-1})_{n \geq 1}$ respectively and let the sequences satisfy (I)–(III) and (1-3). If $C_n = C'_n$ eventually then T and T' are topologically isomorphic.*

Proof. Let X and X' stand for the (C, F) -spaces of T and T' respectively. They are locally compact Cantor spaces. Let $N > 0$ be such that $C_n = C'_n$ for all $n > N$. Then we set $A := [1]_N \subset X$, $A' := [1]_N \subset X'$, $\theta x := x$ for each $x \in A$ and apply Lemma 2.1. \square

We now state and prove one of the main results of this section.

Theorem 2.3. *Let $T = (T_g)_{g \in G}$ and $T' = (T'_g)_{g \in G}$ be two (C, F) -actions of G associated with some sequences $(C_n, F_{n-1})_{n \geq 1}$ and $(C'_n, F'_{n-1})_{n \geq 1}$ respectively and the two sequences satisfy (I)–(III) and (1-3). Then T and T' are topologically isomorphic if and only if there is an increasing sequence of integers*

$$0 = l_0 < l'_1 < l_1 < l'_2 < l_2 < \dots$$

and subsets $A_n \subset F'_{l'_n}$, $B_n \subset F_{l_n}$, such that

$$(2-1) \quad \begin{aligned} A_n B_n &= C_{l_{n-1}+1} \cdots C_{l_n}, \quad B_n A_{n+1} = C'_{l'_n+1} \cdots C'_{l'_{n+1}}, \\ F'_{l'_n} B_n &\subset F_{l_n}, \quad F_{l_n} A_{n+1} \subset F'_{l'_{n+1}}, \\ (F'_{l'_n})^{-1} F'_{l'_n} \cap B_n B_n^{-1} &= F_{l_n}^{-1} F_{l_n} \cap A_{n+1} A_{n+1}^{-1} = \{1\} \end{aligned}$$

for each $n > 0$.

Proof. (\Rightarrow) Let $\phi : X \rightarrow X'$ be a homeomorphism such that $\phi T_g = T'_g \phi$ for each $g \in G$. Then $\phi([1]_0)$ is a clopen subset of X' . Hence there are $l'_1 \geq 0$ and a subset $A_1 \subset F'_{l'_1}$ such that $\phi([1]_0) = [A_1]_{l'_1}$. It follows that

$$\{\phi^{-1}([a]_{l'_1}) \mid a \in A_1\} = \{T_a \phi^{-1}([1]_{l'_1}) \mid a \in A_1\}$$

is a finite partition of $[1]_0$ into compact open subsets. In view of (1-3), there is $l_1 \geq l'_1$ and a subset $B_1 \subset F_{l_1}$ such that

$$\phi^{-1}([1]_{l'_1}) = [B_1]_{l_1}, \quad F'_{l'_1} B_1 \subset F_{l_1}, \quad [1]_0 = [A_1 B_1]_{l_1}$$

and $a B_1 \cap b B_1 = \emptyset$ if $a, b \in F'_{l'_1}$ and $a \neq b$. Since $[1]_0 = [C_1 \cdots C_{l_1}]_{l_1}$, we obtain that $A_1 B_1 = C_1 \cdots C_{l_1}$. In a similar way, it follows from the equality $\phi^{-1}([1]_{l'_1}) = [B_1]_{l_1}$ that there is $l'_2 > l_1$ and a subset $A_2 \subset F'_{l'_2}$ such that

$$\phi([1]_{l_1}) = [A_2]_{l'_2}, \quad F_{l_1} A_2 \subset F'_{l'_2}, \quad B_1 A_2 = C'_{l'_1+1} \cdots C'_{l'_2}$$

and $a A_2 \cap b A_2 = \emptyset$ if $a, b \in F_{l_1}$, and $a \neq b$. Continuing this process infinitely many times, we obtain an increasing sequence $0 = l_0 < l'_1 < l_1 < l'_2 < l_2 < \dots$ and subsets $A_n \subset F'_{l'_n}$ and $B_n \subset F_{l_n}$ for each $n > 0$ such that (2-1) is satisfied.

(\Leftarrow) Suppose that (2-1) is satisfied. Let \tilde{T} and \tilde{T}' denote the (C, F) -actions associated with the $(l_i)_{i \geq 0}$ -telescoping of $(C_n, F_{n-1})_{n \geq 1}$ and the $(l'_i)_{i \geq 1}$ -telescoping of $(C'_n, F'_{n-1})_{n \geq 1}$ respectively. Let \tilde{X} and \tilde{X}' denote the (C, F) -spaces of these actions. Of course, \tilde{T} is isomorphic to T and \tilde{T}' is isomorphic to T' . Take $x \in [1]_0 \subset \tilde{X}$. In view of (2-1), we have a unique expansion $x = (1, a_1 b_1, a_2 b_2, \dots) \in [1]_0$ of x such that $a_i \in A_i$ and $b_i \in B_i$ for all $i \geq 1$. We now set

$$\phi(x) := (a_1, b_1 a_2, b_2 a_3, \dots) \in [A_1]_1 \subset \tilde{X}'.$$

It is standard to verify that ϕ is a homeomorphism of the cylinder $[0]_1 \subset \tilde{X}$ onto the cylinder $[A_1]_1 \subset \tilde{X}'$. Take an element $g \in G$ such that $\tilde{T}_g x \in [1]_0$. Since x and $\tilde{T}_g x$ have the same tails, there is $n > 0$ and $\hat{a}_i \in A_i$, $\hat{b}_i \in B_i$, $i = 1, \dots, n$ such that

$$\tilde{T}_g x = (1, \hat{a}_1 \hat{b}_1, \dots, \hat{a}_n \hat{b}_n, a_{n+1} b_{n+1}, a_{n+2} b_{n+2}, \dots)$$

and hence $g = (\hat{a}_1 \hat{b}_1) \cdots (\hat{a}_n \hat{b}_n) (a_n b_n)^{-1} \cdots (a_1 b_1)^{-1}$. On the other hand,

$$\phi(\tilde{T}_g x) = (\hat{a}_1, \hat{b}_1 \hat{a}_2, \dots, \hat{b}_{n-1} \hat{a}_n, \hat{b}_n a_{n+1}, b_{n+1} a_{n+2}, \dots)$$

and hence $\phi(\tilde{T}_g x) = \tilde{T}_g \phi(x)$. It remains to apply Lemma 2.1. \square

2.2. Topological isomorphism for (C, F) -actions of linearly ordered Abelian groups. We first recall the definition of linearly ordered countable Abelian groups.

Definition 2.4. If A is a countable Abelian group and A_+ is a subset of A such that $A_+ + A_+ \subset A_+$, $A_+ - A_+ = A$ and $A_+ \cap (-A_+) = \{0\}$ then the pair (A, A_+) is called an *ordered group*. If, in addition, $A = A_+ \cup (-A_+)$, we say that (A, A_+) is a *linearly ordered group*.

For instance, $(\mathbb{Z}, \mathbb{Z}_+)$ is a linearly ordered group. More generally, \mathbb{Z}^d endowed with the lexicographical order is a linearly ordered group for each $d > 0$. It was shown in [Le] that an Abelian group admits a linear order if and only if it is torsion free.

In case G is linearly ordered we can strengthen Theorem 2.3.

Theorem 2.5. *Let (G, G_+) be a linearly ordered discrete countable Abelian group. Let $T = (T_g)_{g \in G}$ and $T' = (T'_g)_{g \in G}$ be two (C, F) -actions of G associated with some sequences $(C_n, F_{n-1})_{n \geq 1}$ and $(C'_n, F'_{n-1})_{n \geq 1}$ respectively and the two sequences satisfy (I)–(III) and (1-3). Suppose that $C_n \cup C'_n \subset G_+$ for all n . Then T and T' are topologically isomorphic if and only if there are an increasing sequence of integers $0 = l_0 < l'_1 < l_1 < l'_2 < l_2 < \dots$ and subsets $A_n \subset F'_{l'_n} \cap G_+$ and $B_n \subset F_{l_n} \cap G_+$ such that $0 \in A_n \cap B_n$ and*

$$\begin{aligned} A_n + B_n &= C_{l_{n-1}+1} + \dots + C_{l_n}, & B_n + A_{n+1} &= C'_{l'_n+1} + \dots + C'_{l'_{n+1}}, \\ F'_{l'_n} + B_n &\subset F_{l_n}, & F_{l_n} + A_{n+1} &\subset F'_{l'_{n+1}}, \\ (F'_{l'_n} - F'_{l'_n}) \cap (B_n - B_n) &= (F_{l_n} - F_{l_n}) \cap (A_{n+1} - A_{n+1}) = \{0\} \end{aligned}$$

for each $n > 0$.

Proof. We start with a simple observation. Let A be a finite subset in G . Since G is linearly ordered, there exists $\min A \in G$. Since $0 \in C_n \subset G_+$, we have that

$$\min A = \min(A + C_n) = \min(A + C_n + C_{n+1}) = \dots$$

for each $n > 0$. It follows from this and (1-3) that if $A \subset F_n$ then for each sufficiently large $m > n$, we have that $A' - \min A' \subset F_m$, where $A' := A + C_{n+1} + \dots + C_m$.

In view of that, we can modify in an obvious way the proof of Theorem 2.3 (increasing l_n and l'_n if necessary) so that

$$\tilde{A}_n := A_n - \min A_n \subset F'_{l'_n}, \quad \tilde{B}_n := B_n - \min B_n \subset F'_{l'_n}$$

and the latter two lines of (2-1) hold for \tilde{A}_n and \tilde{B}_n in place of A_n and B_n respectively for each $n > 0$. Since $0 \in \tilde{A}_n$ and

$$\tilde{A}_n + B_n + \min A_n = C_{l_{n-1}+1} + \dots + C_{l_n} \subset G_+$$

by the first line of (2-1) and the condition of the theorem, it follows that $0 \in B_n + \min A_n \subset G_+$. This yields that $\min A_n = -\min B_n$. Hence $\tilde{A}_n + \tilde{B}_n = A_n + B_n = C_{l_{n-1}+1} + \dots + C_{l_n}$. In a similar way, $\min A_{n+1} = -\min B_n$ and $\tilde{B}_n + \tilde{A}_{n+1} = C'_{l'_n+1} + \dots + C'_{l'_{n+1}}$, i.e. the first line in (2-1) holds for \tilde{A}_n and \tilde{B}_n in place of A_n and B_n respectively for each $n > 0$. \square

The conditions for isomorphism of (C, F) -actions are especially simple in the case where T and T' are *commensurate*, i.e. $F_n = F'_n$ eventually.

Theorem 2.6. *Let (G, G_+) be a linearly ordered discrete countable Abelian group. Let $T = (T_g)_{g \in G}$ and $T' = (T'_g)_{g \in G}$ be two (C, F) -actions of G associated with some sequences $(C_n, F_{n-1})_{n \geq 1}$ and $(C'_n, F'_{n-1})_{n \geq 1}$ respectively and the two sequences satisfy (I)–(III) and (1-3). Suppose that $C_n \cup C'_n \subset G_+$ for all n and $F_n = F'_n$ for all $n > N$ (for some $N > 0$). Then T and T' are topologically isomorphic if and only if $C_n = C'_n$ for all $n > M$ (for some $M > 0$).*

Proof. Suppose that T and T' are topologically isomorphic. By Theorem 2.5 and (2-1), for each $n > N$ and $b, \tilde{b} \in B_n$, we have

$$A_n + b \subset F_{l'_n} + b, \quad (F_{l'_n} + b) \cap (F_{l'_n} + \tilde{b}) = \emptyset \quad \text{if } b \neq \tilde{b} \quad \text{and } 0 \in B_n.$$

Therefore

$$A_n = (A_n + B_n) \cap F_{l'_n} = (C_{l_{n-1}+1} + \cdots + C_{l_n}) \cap F_{l'_n} = C_{l_{n-1}+1} + \cdots + C_{l'_n}.$$

This implies, in turn, that

$$(2-2) \quad \bigsqcup_{b \in B_n} (A_n + b) = \bigsqcup_{c \in C_{l'_{n+1}} + \cdots + C_{l_n}} (A_n + c).$$

Hence $\#B_n = \#(C_{l'_{n+1}} + \cdots + C_{l_n})$. Moreover, (2-2) yields that there are two maps $B_n \ni b \mapsto \gamma(b) \in C_{l'_{n+1}} + \cdots + C_{l_n}$ and $B_n \ni b \mapsto \alpha(b) \in A_n$ such that $b = \alpha(b) + \gamma(b)$ for each $b \in B_n$. We first note that the map γ is one-to-one. Indeed, if $\gamma(b_1) = \gamma(b_2)$ then $b_1 - \alpha(b_1) = b_2 - \alpha(b_2)$ and hence $b_1 - b_2 \in F_{l'_n} - F_{l'_n}$. The latter implies that $b_1 = b_2$, as desired. Hence γ is a bijection of B_n onto $C_{l'_{n+1}} + \cdots + C_{l_n}$ (we recall that these finite sets are of the same cardinality). Comparing the sums of the elements in the lefthand side and the righthand side of (2-2) we obtain that

$$\#B_n \sum_{a \in A_n} a + \sum_{b \in B_n} (\alpha(b) + \gamma(b)) = \#(C_{l'_{n+1}} + \cdots + C_{l_n}) \sum_{a \in A_n} a + \sum_{c \in C_{l'_{n+1}} + \cdots + C_{l_n}} c.$$

Hence $\sum_{b \in B_n} \alpha(b) = 0$. Since $\alpha(b) \in A_n \subset G_+$, it follows that $\alpha(b) = 0$ for each $b \in B_n$. Hence $B_n = C_{l'_{n+1}} + \cdots + C_{l_n}$. In a similar way we derive from the second equality in the first line of (2-1) that $B_n = C'_{l'_{n+1}} + \cdots + C'_{l'_n}$ and $A_{n+1} = C'_{l_{n+1}} + \cdots + C'_{l'_{n+1}}$ for all sufficiently large n . Therefore

$$\begin{aligned} C_{l'_{n+1}} + \cdots + C_{l_n} &= C'_{l'_{n+1}} + \cdots + C'_{l'_n} \quad \text{and} \\ C_{l_{n+1}} + \cdots + C_{l'_{n+1}} &= C'_{l_{n+1}} + \cdots + C'_{l'_{n+1}}. \end{aligned}$$

It follows now from (III) and the equality $F_j = F'_j$ eventually that $C_j = C'_j$ eventually, as claimed.

The converse implication is obvious. \square

2.3. Applications to topological centralizers and inverse actions. Given a topological action T of G on a locally compact Cantor space X , we let

$$C_{\text{top}}(T) := \{\theta \in \text{Homeo}(X) \mid \theta T_g = T_g \theta \text{ for each } g \in G\}$$

and call this set *the topological centralizer* of T .

Corollary 2.7. *Let (G, G_+) be a linearly ordered discrete countable Abelian group. Let T be a (C, F) -action of G associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(III) and (1-3). If $C_n \subset G_+$ for each $n \geq 1$ then $C_{\text{top}}(T) = \{T_g \mid g \in G\}$.*

Proof. Let $\theta \in C_{\text{top}}(X)$. Using ϕ we can construct the sequences $0 = l_0 < l'_1 < l_1 < l'_2 < l_2 < \dots$ and $(A_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$ as in the proof of Theorem 2.3. In particular, $\phi([0]_{l_n}) = [A_{n+1}]_{l'_{n+1}}$ for each n . It was shown in the proof of Theorem 2.5 that $\min A_n = -\min B_n = \min A_{n+1}$ for each $n > 0$. Hence there is $g_0 \in G$ such that

$$g_0 = \min A_1 = \min A_2 = \dots = -\min B_1 = -\min B_2 = \dots.$$

We now set $\psi := T_{-g_0} \circ \phi$. Then

$$\psi([0]_{l_n}) = T_{-g_0}[A_{n+1}]_{l'_{n+1}} = [A_{n+1} - g_0]_{l'_{n+1}} = [A_{n+1} - \min A_{n+1}]_{l'_{n+1}} \supset [0]_{l'_{n+1}}$$

for each n . We let $\mathbf{0} := (0, 0, \dots) \in X_0 \subset X$. Hence

$$\psi(\mathbf{0}) = \psi\left(\bigcap_{n=1}^{\infty} [0]_{l_n}\right) = \bigcap_{n=1}^{\infty} \psi([0]_{l_n}) \supset \bigcap_{n=1}^{\infty} [0]_{l'_{n+1}} = \mathbf{0}.$$

It follows that $\psi(\mathbf{0}) = \mathbf{0}$. Since ψ is equivariant, we obtain that $\psi(x) = x$ for each x from the T -orbit of $\mathbf{0}$. Thus ψ and Id coincide on a dense subset of X . Therefore $\psi = \text{Id}$ and hence $\phi = T_{g_0}$. \square

We note that, in particular, the odometers have trivial (topological) centralizer in their locally compact (C, F) -realizations.

Given a continuous action $T = (T_g)_{g \in G}$ on an Abelian group G on a topological space X , we let $T^{-1} := (T_{-g})_{g \in G}$. Then T^{-1} is also a continuous action of G on X . We call it the *inverse to T* .

Lemma 2.8. *Let (G, G_+) be a linearly ordered discrete countable Abelian group. Let T be a (C, F) -action of G associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(III) and (1-3). Then T^{-1} is (topologically) isomorphic a (C, F) -action associated with the sequence $(C_n^*, F_{n-1}^*)_{n \geq 1}$, where $C_n^* := \{-c + \max C_n \mid c \in C_n\}$ and $F_n^* := \{-f + \sum_{j=1}^n \max C_j \mid f \in F_n\}$.*

Sketch of the proof. It is straightforward to verify that (I)–(III) and (1-3) are satisfied for the sequence $(C_n^*, F_{n-1}^*)_{n \geq 1}$. The canonical isomorphism of T with T^{-1} is given by the map

$$(f_n, c_{n+1}, \dots) \mapsto \left(\sum_{j=1}^n \max C_j - f_n, \max C_{n+1} - c_{n+1}, \max C_{n+2} - c_{n+2}, \dots \right)$$

from $X_n := F_n \times C_{n+1} \times C_{n+2} \times \dots$ onto $X_n^* := F_n^* \times C_{n+1}^* \times C_{n+2}^* \times \dots$ for each $n \geq 0$.

Corollary 2.9. *Let (G, G_+) be a linearly ordered discrete countable Abelian group. Let T be a (C, F) -action associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(III) and (1-3). Suppose that $\bigcup_{n \geq 1} C_n \subset G_+$ and that (II) and (1-3) are satisfied for the sequence $(C_n^*, F_{n-1}^*)_{n \geq 1}$. Then T is topologically isomorphic to T^{-1} if and only if $C_n = C_n^*$ eventually.*

Proof. Of course, (I) and (III) are also satisfied for $(C_n^*, F_{n-1})_{n \geq 1}$. Hence a (C, F) -action associated with $(C_n^*, F_{n-1})_{n \geq 1}$ is well defined. By Corollary 2.2, this action is isomorphic to T^{-1} . Now Theorem 2.6 yields that T and T^{-1} are isomorphic if and only if $C_n = C_n^*$ eventually. \square

Remark 2.10. We note that the conditions of Corollary 2.9 are satisfied for the most important example where $(G, G_+) = (\mathbb{Z}, \mathbb{Z}_+)$ and every set F_n is an *order interval* $\{j \in \mathbb{Z} \mid -\alpha_n \leq j \leq \beta_n\}$ for some integers $\alpha_n, \beta_n > 0$. This follows from the following simple fact: if A is a finite subset of \mathbb{Z} such that $\min A \in F_n$ and $\max A \in F_n$ then $A \subset F_n$. It remains to notice that $\min C_n = \min C_n^* = 0$ and $\max C_n = \max C_n^*$ for each $n > 0$.

3. (C, F) -MODELS FOR MEASURE PRESERVING ACTIONS OF MONOTILEABLE AMENABLE GROUPS

Let G be a *monotileable* amenable discrete countable group [We]. This means that there is a Følner sequence $(\mathcal{F}_n)_{n \geq 0}$ in G such that every set \mathcal{F}_n tiles G , i.e. there is a subset $\mathcal{C}_n \subset G$ such that $\mathcal{F}_n \mathcal{C}_n = G$ and $\mathcal{F}_n c \cap \mathcal{F}_n c' = \emptyset$ whenever $c, c' \in \mathcal{C}_n$ and $c \neq c'$. Without loss of generality we may assume that $1 \in \mathcal{F}_n \cap \mathcal{C}_n$ for each $n \geq 0$.

Fix a standard nonatomic probability space (X, \mathfrak{B}, μ) . Denote by $\text{Aut}(X, \mu)$ the group of μ -preserving transformations. It is a Polish group when endowed with the weak operator topology. Let \mathcal{A}_G stand for the set of μ -preserving actions of G on X . An element of \mathcal{A}_G is a group homomorphism from G to $\text{Aut}(X, \mu)$. Thus \mathcal{A}_G is a subset of the infinite product space $\text{Aut}(X, \mu)^G$. Endow the latter space with the infinite product of the weak operator topologies on $\text{Aut}(X, \mu)$. Then $\text{Aut}(X, \mu)^G$ is a Polish space and \mathcal{A}_G is a closed subset of it. Hence \mathcal{A}_G is Polish in the induced topology.

Let \mathfrak{F} denote the set of all finite subsets of G . We now let

$$\mathfrak{R}_1 := \{(C_n, F_{n-1})_{n \geq 1} \in (\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}} \mid (F_n)_{n \geq 0} \text{ is a subsequence of } (\mathcal{F}_n)_{n \geq 0} \text{ and (I)–(III) and (1-3) are satisfied}\}.$$

Endow \mathfrak{F} with the discrete topology. Then the space $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$ furnished with the infinite product topology—we denote it by τ —is a Polish 0-dimensional space.

Lemma 3.1. *The subset \mathfrak{R}_1 is a G_δ -subset of $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$.*

Proof. It is clear that each of the conditions (I)–(III) determines a closed subset in $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$. The condition (1-3) determines a G_δ -subset in $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$. \square

Hence \mathfrak{R}_1 endowed with τ is Polish and 0-dimensional. The function

$$\phi_m : \mathfrak{R}_1 \ni (C_n, F_{n-1})_{n \geq 1} \mapsto \frac{\#F_m}{\#C_1 \cdots \#C_m} \in \mathbb{R}$$

is continuous on \mathfrak{R}_1 for each $m > 0$. Moreover, $\phi_1 \leq \phi_2 \leq \cdots$. Hence the extended function $\phi := \sup_{n > 0} \phi_n$ taking values in $\mathbb{R} \cup \{+\infty\}$ is lower semicontinuous. We note that $\phi((C_n, F_{n-1})_{n \geq 1}) = \lim_{m \rightarrow \infty} \frac{\#F_m}{\#C_1 \cdots \#C_m}$. We now set

$$\mathfrak{R}_1^{\text{fin}} := \{S \in \mathfrak{R}_1 \mid \phi(S) < \infty\}.$$

Then $\mathfrak{R}_1^{\text{fin}}$ is an F_σ -subset of \mathfrak{R}_1 .

Definition 3.2. Denote by τ^{fin} the weakest topology on $\mathfrak{R}_1^{\text{fin}}$ which is stronger than τ and such that ϕ is continuous in it.

It is well known that τ^{fin} is Polish (see, e.g. [Ke]). We note that a sequence $(\mathcal{S}_n)_{n=1}^\infty$ in $\mathfrak{R}_1^{\text{fin}}$ converges in τ^{fin} to an element $\mathcal{S} \in \mathfrak{R}_1^{\text{fin}}$ if and only if $(\mathcal{S}_n)_{n=1}^\infty$ converges in τ to \mathcal{S} as $n \rightarrow \infty$ and there is $\lim_{n \rightarrow \infty} \phi(\mathcal{S}_n) = \phi(\mathcal{S})$.

Our purpose now is to construct a continuous map

$$\Psi : \mathfrak{R}_1^{\text{fin}} \ni \mathcal{S} \mapsto \Psi^{\mathcal{S}} \in \mathcal{A}_G.$$

Without loss of generality we may assume that $X = [0, 1)$ and μ is the Lebesgue measure on $[0, 1)$. Fix a linear order \succ on G . Given $\mathcal{S} := (C_n, F_{n-1})_{n \geq 1} \in \mathfrak{R}_1^{\text{fin}}$, we denote by $T^{\mathcal{S}} = (T_g^{\mathcal{S}})_{g \in G}$ the (C, F) -action of G associated with \mathcal{S} . Let $X^{\mathcal{S}}$ and $\mu^{\mathcal{S}}$ stand for the (C, F) -space and the (C, F) -measure of $T^{\mathcal{S}}$ respectively. For each $n > 0$, we set

$$\alpha_0 := \frac{1}{\phi(\mathcal{S})} \quad \text{and} \quad \alpha_n := \frac{\alpha_0 \# F_n}{\# C_1 \cdots \# C_n}.$$

Then the sequence $(\alpha_n)_{n=1}^\infty$ increases and converges to 1 as $n \rightarrow \infty$. For each $n \geq 0$, we partition the interval $[0, \alpha_n)$ into subintervals $I_f^{(n)}$, $f \in F_n$, of length $\alpha_n / \# F_n$ such that

- every subinterval $I_f^{(n)}$, $f \in F_n$, is the union of pairwise disjoint subintervals $I_{fc}^{(n+1)}$, $c \in C_{n+1}$,
- if $f \in F_n$, $c, c' \in C_{n+1}$ and $c \succ c'$ then $I_{fc}^{(n)}$ is on the right of $I_{f'c'}^{(n)}$,
- if $f, f' \in F_{n+1} \setminus (F_n C_{n+1})$ and $f \succ f'$ then $I_f^{(n)}$ is on the right of $I_{f'}^{(n)}$.

It is obvious that these conditions determine the partition $[0, \alpha_n) = \bigsqcup_{g \in F_n} I_g^{(n)}$ and the enumeration of its atoms by elements of F_n in a unique way for each $n \geq 0$. By the standard properties of Lebesgue spaces, there is a unique (mod 0) Borel bijection $\beta : X^{\mathcal{S}} \rightarrow [0, 1)$ such that $\beta([f]_n) = I_f^{(n)} \pmod{0}$ for each $f \in F_n$ and $n \geq 0$ and $\mu^{\mathcal{S}}(X^{\mathcal{S}})^{-1} \mu^{\mathcal{S}} = \mu \circ \beta$. We now define the G -action $\Psi^{\mathcal{S}} = (\Psi_g^{\mathcal{S}})_{g \in G}$ on X by setting $\Psi_g^{\mathcal{S}} := \beta T_g^{\mathcal{S}} \beta^{-1}$, $g \in G$. Thus, if $g \in G$ and $f \in F_n$ with $gf \in F_n$ then $\Psi_g^{\mathcal{S}} I_f^{(n)} = I_{gf}^{(n)}$, $n \in \mathbb{N}$.

Lemma 3.3. Ψ is continuous.

Proof. If $\mathcal{S} = (C_n, F_{n-1})_{n \geq 1}$ and $\mathcal{S}' = (C'_n, F'_{n-1})_{n \geq 1}$ are τ^{fin} -close, there is $\epsilon > 0$ and $N > 0$ such that $(C_n, F_{n-1}) = (C'_n, F'_{n-1})$ for each $n = 1, \dots, N$ and $\phi(\mathcal{S}) = (1 \pm \epsilon)\phi(\mathcal{S}')$. Let $I_f^{(n)}$, $f \in F_n$, denote the subintervals of $[0, 1)$ used in the definition of $\Psi^{\mathcal{S}}$ and let $J_f^{(n)}$, $f \in F_n$, denote the similar subintervals of $[0, 1)$ used to define $\Psi^{\mathcal{S}'}$ for $n = 0, 1, \dots, N$. Then for each $n = 1, \dots, N - 1$ and $f \in F_n$, we have $\mu(I_f^{(n)} \triangle J_f^{(n)}) \leq 2\epsilon\mu(I_f^{(n)})$. Hence for each $g \in G$ and $f \in F_n$ such that $gf \in F_n$, we obtain that

$$\mu(\Psi_g^{\mathcal{S}} I_f^{(n)} \triangle \Psi_g^{\mathcal{S}'} I_f^{(n)}) \leq \mu(I_{gf}^{(n)} \triangle J_{gf}^{(n)}) + 2\epsilon\mu(I_f^{(n)}) \leq 4\epsilon\mu(I_f^{(n)}).$$

A standard argument yields now that $\Psi^{\mathcal{S}}$ and $\Psi^{\mathcal{S}'}$ are close in \mathcal{A}_G if ϵ is small enough and N is large enough. \square

Denote by \mathbf{R} the *tail equivalence relation* on $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$. This means that two sequences $(D_n)_{n \geq 1}$ and $(D'_n)_{n \geq 1}$ from $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$ are \mathbf{R} -equivalent if $D_n = D'_n$ eventually. It is easy to see that \mathbf{R} is an F_σ -subset of $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}} \times (\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$. It is straightforward to verify that each \mathbf{R} -class is dense in $(\mathfrak{F} \times \mathfrak{F})^{\mathbb{N}}$. Of course, the subsets \mathfrak{R}_1 and $\mathfrak{R}_1^{\text{fin}}$ are \mathbf{R} -saturated. Hence \mathbf{R} restricted to $\mathfrak{R}_1^{\text{fin}}$ is also F_σ in τ^{fin} . It is easy to see that if $\mathcal{S}, \mathcal{S}' \in \mathfrak{R}_1^{\text{fin}}$ and $(\mathcal{S}, \mathcal{S}') \in \mathbf{R}$ then the G -actions $\Psi^{\mathcal{S}}$ and $\Psi^{\mathcal{S}'}$ are isomorphic.

Lemma 3.4. *For each $\mathcal{S} \in \mathfrak{R}_1^{\text{fin}}$, the \mathbf{R} -class of \mathcal{S} is τ^{fin} -dense in $\mathfrak{R}_1^{\text{fin}}$.*

Proof. Let $\mathcal{S} = (C_n, F_{n-1})_{n \geq 1}$. Take an arbitrary $\widehat{\mathcal{S}} = (\widehat{C}_n, \widehat{F}_{n-1})_{n \geq 1} \in \mathfrak{R}_1^{\text{fin}}$. We will construct $\mathcal{S}' \in \mathfrak{R}_1^{\text{fin}}$ which is τ^{fin} -close (as close as we wish) to $\widehat{\mathcal{S}}$ and such that $(\mathcal{S}', \mathcal{S}) \in \mathbf{R}$.

Fix $\epsilon > 0$. Select $j > 0$ such that $\phi_j(\widehat{\mathcal{S}}) > \phi(\widehat{\mathcal{S}}) - \epsilon$. For $i > 0$, let

$$F_i^\circ := \{f \in F_i \mid \widehat{F}_j \widehat{F}_j^{-1} f \subset F_i\}.$$

Since $(F_n)_{n \geq 1}$ is a Følner sequence in G , there exists $i > 0$ such that $\#F_i^\circ > (1 - \epsilon)\#F_i$. Since $\phi(\mathcal{S}) < \infty$, we can also assume without loss of generality that

$$(3-1) \quad \frac{\#F_m}{\#F_i \#C_{i+1} \cdots \#C_m} < 1 + \epsilon$$

for each $m > i$. Let $C := \{c \in C_j \mid F_i^\circ \cap \widehat{F}_j c \neq \emptyset\}$. Then $F_i \supset \widehat{F}_j C \supset F_i^\circ$ and $\widehat{F}_j c \cap \widehat{F}_j c' = \emptyset$ if $c, c' \in C$ and $c \neq c'$. It follows that

$$(3-2) \quad \#F_i < (1 + \epsilon)\#C\#\widehat{F}_j.$$

We now set

$$(3-3) \quad (C'_{a+1}, F'_a) := \begin{cases} (\widehat{C}_{a+1}, \widehat{F}_a) & \text{if } a < j, \\ (C, \widehat{F}_a) & \text{if } a = j, \\ (C_{a-j+i}, F_{a-j+i-1}) & \text{if } a > j. \end{cases}$$

Let $\mathcal{S}' := (C'_{a+1}, F'_a)_{a \geq 0}$. It is obvious that (I)–(III) are satisfied for \mathcal{S}' for each $a > 0$. We note that (2-3) holds for \mathcal{S}' because it holds for \mathcal{S} . Hence $\mathcal{S}' \in \mathfrak{R}_1$. For each $m > j$, we have

$$\phi_m(\mathcal{S}') := \frac{\#F'_m}{\#C'_1 \cdots \#C'_m} = \phi_j(\widehat{\mathcal{S}}) \cdot \frac{\#F_i}{\#C\#\widehat{F}_j} \cdot \frac{\#F_{m+i-j}}{\#F_i \#C_{i+1} \cdots \#C_{m+i-j}}.$$

Applying (3-1) and (3-2) we obtain that $\phi_m(\mathcal{S}') = (\phi(\mathcal{S}) \pm \epsilon)(1 \pm \epsilon)^2$. It follows that $\mathcal{S}' \in \mathfrak{R}_1^{\text{fin}}$ and $\phi(\mathcal{S}')$ is close to $\phi(\mathcal{S})$. The first line in (3-3) yields that \mathcal{S}' is τ -close to \mathcal{S} (if j is chosen large). Hence \mathcal{S}' is τ^{fin} -close to \mathcal{S} . It follows from the third line in (3-3) that $(\mathcal{S}', \mathcal{S}) \in \mathbf{R}$, as desired. \square

We now generalize the concept of model for $\text{Aut}(X, \mu)$ (viewed as the \mathbb{Z} -actions on (X, μ)) introduced in [Fo, Definition 10] to the G -actions on (X, μ) .

Definition 3.5. A *model* for \mathcal{A}_G is a pair (W, π) , where W is a Polish space and $\pi : W \rightarrow \mathcal{A}_G$ is a continuous map such that for a comeager set $\mathcal{M} \subset \mathcal{A}_G$ and each $A \in \mathcal{M}$, the set $\{w \in W \mid \pi(w) \text{ is isomorphic to } A\}$ is dense in W .

We let $\mathcal{O}_G := \{T \in \mathcal{A}_G \mid T \text{ is of rank one along } (\mathcal{F}_n)_{n \geq 0}\}$.

Proposition 3.6. \mathcal{O}_G is a dense G_δ (and hence comeager) in \mathcal{A}_G .

Proof. We first prove that \mathcal{O}_G is a G_δ . Without loss of generality we may assume that X is a compact Cantor space. Let \mathcal{K} denote the class of clopen subsets in X . Since \mathcal{K} is countable, we can write it as $\mathcal{K} = \{K_i \mid i \in \mathbb{N}\}$. Given $A \in \mathcal{K}$, a finite subset $F \subset G$, $m > 0$ and an action $T = (T_g)_{g \in G} \in \mathcal{A}_G$, we set

$$\alpha_{m,A,F}(T) := \max_{1 \leq i \leq m} \min_{F' \subset F} \mu \left(K_i \triangle \bigcup_{g \in F'} T_g A \right)$$

$$\beta_{A,F}(T) := \max_{g \in F} \mu(T_g A \cap A)$$

Then $\alpha_{m,A,F} : \mathcal{A}_G \ni T \mapsto \alpha_{m,A,F}(T) \in \mathbb{R}$ and $\beta_{A,F} : \mathcal{A}_G \ni T \mapsto \beta_{A,F}(T) \in \mathbb{R}$ are continuous functions. It is straightforward to verify that \mathcal{O}_G equals

$$\bigcap_{r=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{s>r} \bigcup_{A \in \mathcal{K}} \left\{ T \in \mathcal{A}_G \mid \alpha_{m,A,\mathcal{F}_s}(T) < \frac{\mu(A)}{l \# \mathcal{F}_s} \text{ and } \beta_{A,\mathcal{F}_s}(T) < \frac{\mu(A)}{l \# \mathcal{F}_s} \right\}.$$

Hence it is a G_δ .

By [FoWe, Claim 18] (see also [Da3, just below Proposition 1.2] for the proof), given a free $T \in \mathcal{A}_G$, the class of G -actions on (X, μ) that are isomorphic to T is dense in \mathcal{A}_G . Hence it remains to show that \mathcal{O}_G contains a free G -action. For that we will utilize the (C, F) -construction. As in the proof of Lemma 3.4, we can construct inductively the sequence $(C_n, F_{n-1})_{n \geq 1}$ such that $F_n = \mathcal{F}_{l_n}$, $C_{n+1} \subset \mathcal{C}_{l_n}$ for some increasing sequence $l_1 < l_2 < \dots$ and the conditions (I)–(III), (1-2) and (1-3) are satisfied. Then the associated (C, F) -action of G belongs to \mathcal{O}_G . It is free because every (C, F) -action is free. \square

Since Ψ takes values in \mathcal{O}_G , we deduce from Lemmata 3.3 and 3.4 and Proposition 3.6 the following corollary.

Corollary 3.7. $(\mathfrak{R}_1^{\text{fin}}, \Psi)$ is a model for \mathcal{A}_G .

4. MEASURE PRESERVING RANK-ONE ACTIONS WITH BOUNDED PARAMETERS

In this section G is Abelian. We extend here the main results from [GaHi3].

4.1. Rigidity and rigidity along asymptotically invariant subsequence of subsets. Let $S = (S_g)_{g \in G}$ stand for an ergodic free action of G on a standard probability measure space (Y, ν) . S is called *rigid* if there is a sequence $(g_n)_{n=1}^{\infty}$ of elements from G such that $g_n \rightarrow \infty$ and $S_{g_n} \rightarrow \text{Id}$ as $n \rightarrow \infty$ in the Polish group $\text{Aut}(Y, \nu)$. The sequence $(g_n)_{n=1}^{\infty}$ is called a *rigidity sequence* for S . It follows straightforwardly that

- for each $0 \neq p \in \mathbb{Z}$, the sequence $(pg_n)_{n=1}^{\infty}$ is also a rigidity sequence for S ;
- if $(g'_n)_{n=1}^{\infty}$ is another rigidity sequence in G then either $g_n = g'_n$ eventually or the difference $(g_n - g'_n)_{n=1}^{\infty}$ is a rigidity sequence for S ;
- a subsequence of the rigidity sequence for S is a rigidity sequence for S .

We need a generalization of the rigidity concept. Let $(W_n)_{n=1}^\infty$ be an *asymptotically S -invariant* sequence of subsets in Y , i.e. $\nu(S_g W_n \Delta W_n) \rightarrow 0$ for each $g \in G$ and $\mu(W_n) \rightarrow \delta > 0$ as $n \rightarrow \infty$. It is a standard fact that then $\nu(A \cap W_n) \rightarrow \delta \nu(A)$ for each subset $A \subset Y$ (see, for instance, [Da6, Lemma 4.6] for the case where $G = \mathbb{Z}$; in the general case the proof is similar).

The following definition appeared in [BeFr] in the case $G = \mathbb{Z}$.

Definition 4.1. If there is a sequence $(g_n)_{n=1}^\infty$ of elements from G such that for each subset $A \subset Y$,

$$(4-1) \quad \nu((S_{g_n} A \Delta A) \cap W_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

then S is called *rigid* along $(W_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ is called a *rigidity sequence* for S along $(W_n)_{n=1}^\infty$.

We note that (4-1) is equivalent to

$$(4-2) \quad \nu(S_{g_n} A \cap A \cap W_n) \rightarrow \delta \mu(A) \quad \text{as } n \rightarrow \infty$$

Of course, each rigidity sequence for S is a rigidity sequence for S along every asymptotically S -invariant sequence.

Lemma 4.2. Let $(g_n)_{n=1}^\infty$ be a rigidity sequence for S along $(W_n)_{n=1}^\infty$. If $(g+g_n)_{n=1}^\infty$ is also a rigidity sequence for S along $(W_n)_{n=1}^\infty$ for some $g \in G$ then $g = 0$.

Proof. For each subset $A \subset Y$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \nu((S_g S_{g_n} A \Delta A) \cap W_n) &= \lim_{n \rightarrow \infty} \nu((S_{g_n} A \Delta S_{-g} A) \cap S_{-g} W_n) \\ &= \lim_{n \rightarrow \infty} \nu((S_{g_n} A \Delta S_{-g} A) \cap W_n). \end{aligned}$$

It follows from this, (4-1) and the condition of the lemma that

$$\lim_{n \rightarrow \infty} \nu((A \Delta S_{-g} A) \cap W_n) = 0.$$

Since $(W_n)_{n=1}^\infty$ is asymptotically S -invariant, we obtain that $\nu((A \Delta S_{-g} A)) = 0$. Hence S_{-g} is the identity and we are done. \square

4.2. Bounded (C, F) -constructions. Rigidity criterium for the bounded rank-one actions of \mathbb{Z} . Let $T = (T_j)_{j \in \mathbb{Z}}$ be a (C, F) -action of G associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(III) and (1-4). The following definition generalizes naturally Ryzhikov's definition given in the case of rank-one actions of \mathbb{Z} [Ry].

Definition 4.3. We say that the sequence of parameters $(C_n, F_{n-1})_{n \geq 1}$ is *bounded* if there is $R > 0$ and a finite subset $K \subset G$ such that $\#C_n \leq R$ and $K + F_n + C_{n+1} \supset F_{n+1}$ for each $n \geq 0$.

It is easy to see that each (C, F) -action associated with bounded sequence of parameters is *finite* measure preserving. We note also that if a sequence $(\tilde{C}_n, \tilde{F}_{n-1})_{n \geq 0}$ is a bounded telescoping of a bounded sequence $(C_n, F_{n-1})_{n \geq 1}$ then $(\tilde{C}_n, \tilde{F}_{n-1})_{n \geq 0}$ is also bounded. In particular, for each $p > 0$, the $(pn)_{n \geq 0}$ -telescoping of $(C_n, F_{n-1})_{n \geq 1}$ yields a bounded sequence of parameters. \blacksquare

From now on let $G = \mathbb{Z}$ and $F_n = \{0, 1, \dots, h_n - 1\}$ for some integers $h_n \geq 0$. In other words, T is a rank-one action of \mathbb{Z} . Then $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$ for certain r_n and σ_n via (1-5) and (1-6) for each $n \geq 0$. We note that $(C_n, F_{n-1})_{n \geq 1}$ is bounded if and only if there is $R > 0$ such that $r_n \leq R$ and $\max_{1 \leq i \leq r_n} \sigma_n(i) \leq R$ for each $n > 0$. We also note that if $(C_n, F_{n-1})_{n \geq 1}$ is bounded then the canonical measure μ is finite, i.e. (1-2) is satisfied.

We need a notation. Given two maps

$$a : \{1, \dots, r\} \ni i \mapsto a(i) \in \mathbb{Z}_+ \quad \text{and} \quad b : \{1, \dots, t\} \ni i \mapsto b(i) \in \mathbb{Z}_+,$$

we define a new map $a \diamond b : \{1, \dots, rt\} \ni i \mapsto a \diamond b \in \mathbb{Z}_+$ by setting

$$a \diamond b(i) = \begin{cases} a(j) & \text{if } i \equiv j \pmod{r} \text{ and } 1 \leq j < r, \\ a(r) + b(k) & \text{if } i = rk \text{ and } 1 \leq k \leq t. \end{cases}$$

Our interest to this “operation” is due to the following property of the parameters of rank-one (C, F) -actions of \mathbb{Z} : if $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$ and $(C_{n+2}, F_{n+2}) \sim (r_{n+1}, \sigma_{n+1})$ then $(C_{n+1} + C_{n+2}, F_{n+2}) \sim (r_n r_{n+1}, \sigma_n \diamond \sigma_{n+1})$ for each $n \in \mathbb{N}$. We leave verification of this property to the readers because it is a routine.

Theorem 4.4. *Let T be a (C, F) -action of \mathbb{Z} associated with a bounded sequence $(C_n, F_{n-1})_{n \geq 1}$ and let $F_n = \{0, 1, \dots, h_n - 1\}$ for some integers $h_n \geq 1$ for each $n \geq 0$. Then T is rigid if and only if for each $N > 0$, there are integers n, m such that $m > n + N > n > N$ and the set $C_n + \dots + C_m$ is an arithmetic sequence.*

Proof. The “if” part is trivial. As for the “only if”, we first note that it is enough to prove the following claim:

- for each $N > 0$, there is $n > N$ such that C_n is an arithmetic sequence.

Indeed, if this claim is true then for each $k > 0$, the action T is associated also with a $(kn)_{n=1}^\infty$ -telescoping of $(C_n, F_{n-1})_{n \geq 1}$. The telescoping yields a bounded sequence of parameters. It remains to apply the claim to this sequence of parameters.

We now prove the claim. Assume that T is rigid. Let $(F_{n+1}, C_{n+1}) \sim (r_n, \sigma_n)$ for each $n \geq 0$. Since $(C_n, F_{n-1})_{n \geq 1}$ is bounded, there is $R > 0$ such that $\sup_{n \geq 1} r_n \leq R$ and $\sup_{n \geq 1} \max_{1 \leq i \leq r_n} \sigma_n(i) \leq R$. Passing to the $(2n)_{n=1}^\infty$ -telescoping we may assume without loss of generality that $\#C_n \geq 4$ for each $n > 0$. Choose a rigidity sequence $(g_n)_{n=1}^\infty$ for T such that $g_n \rightarrow +\infty$ as $n \rightarrow \infty$. Then for each $n > 0$, there is a unique $l_n > 0$ such that $g_n \in F_{l_n+1} \setminus F_{l_n}$. In view of the boundedness condition we may assume without loss of generality that there exists $f_n \in F_{l_n} + \{0, 1, \dots, R^2\}$ and $i_n \in \{1, \dots, r_{l_n} - 1\}$ such that $g_n = f_n + i_n h_{l_n}$. Using again the boundedness of $(C_n, F_{n-1})_{n \geq 1}$ and passing (if necessary) to a further subsequence of $(g_n)_{n=1}^\infty$ we may assume in addition that there are integers $r, r' \leq R$, $i \in \{1, \dots, r - 1\}$, a real $\delta \in [0, 1]$ and maps $\sigma : \{1, \dots, r\} \rightarrow \{0, 1, \dots, R\}$ and $\sigma' : \{1, \dots, r'\} \rightarrow \{0, 1, \dots, R\}$ such that $r_{l_n} = r$, $r_{l_n+1} = r'$, $i_n = i$, $\sigma_{l_n} = \sigma$ and $\sigma_{l_n+1} = \sigma'$ for all $n > 0$ and $\lim_{n \rightarrow \infty} f_n/h_{l_n} = \delta$. Let s stand for the integral of σ . We consider separately three cases.

Case A. Let $\delta < 1$. Take a cylinder A in X . Then for each sufficiently large n , there is a subset $A_n \subset F_n$ such that $A = [A_n]_n$. We now set

$$W_n := [F_{l_n} \cap (F_{l_n} - f_n - 2R^2)]_{l_n} \quad \text{and} \\ V_n := \left[\bigcup_{j=0}^{r-1-i} (F_{l_n} \cap (F_{l_n} - f_n - 2R^2) + j h_{l_n} + s(j)) \right]_{l_n+1}.$$

It is easy to verify that the two sequences $(W_n)_{n=1}^\infty$ and $(V_n)_{n=1}^\infty$ are asymptotically T -invariant,

$$(4-3) \quad \lim_{n \rightarrow \infty} \mu(W_n) = 1 - \delta > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mu(V_n) = \frac{(1 - \delta)(r - i)}{r} > 0.$$

We set $A_{l_n}^\circ := A_{l_n} \cap (F_{l_n} - f_n - 2R^2)$. Since $g_n = f_n + ih_{l_n}$, it follows that

$$T_{g_n}(A \cap V_n) = \bigcup_{j=i}^{r-1} [(A_{l_n}^\circ + f_n + s(j - i) - s(j)) + jh_{l_n} + s(j)]_{l_n+1},$$

because $A_{l_n}^\circ + f_n + s(j - i) - s(j) \subset F_{l_n}$ for each $j = i, \dots, r - 1$. Therefore

$$\begin{aligned} \mu(T_{g_n}(A \cap V_n) \cap A) &= \frac{1}{r} \sum_{j=i}^{r-1} \mu([(s(j - i) - s(j) + f_n + A_{l_n}^\circ) \cap A_{l_n}]_{l_n}) \\ &= \frac{1}{r} \sum_{j=i}^{r-1} \mu([(s(j - i) - s(j) + f_n + A_{l_n}^\circ)]_{l_n} \cap [A_{l_n}]_{l_n}) \\ &= \frac{1}{r} \sum_{j=i}^{r-1} \mu(T_{s(j-i)-s(j)+f_n}[A_{l_n}^\circ]_{l_n} \cap A) \\ &= \frac{1}{r} \sum_{j=i}^{r-1} \mu(T_{s(j-i)-s(j)+f_n}(A \cap W_n) \cap A). \end{aligned}$$

Passing to the limit and utilizing (4-3) and the rigidity of $(g_n)_{n \geq 1}$ we obtain that

$$\begin{aligned} \frac{(1 - \delta)(r - i)\mu(A)}{r} &= \lim_{n \rightarrow \infty} \mu((A \cap V_n) \cap T_{-g_n}A) \\ &= \lim_{n \rightarrow \infty} \frac{1}{r} \sum_{j=i}^{r-1} \mu(A \cap W_n \cap T_{s(j)-s(j-i)-f_n}A) \\ &\leq \frac{(1 - \delta)(r - i)\mu(A)}{r}. \end{aligned}$$

The equality is only possible if for each $j = i, \dots, r - 1$, there exists a limit

$$\lim_{n \rightarrow \infty} \mu(A \cap W_n \cap T_{s(j)-s(j-i)-f_n}A) = (1 - \delta)\mu(A).$$

Applying the standard approximation argument we conclude that the limit exists for an arbitrary Borel set $A \subset X$, not only for the cylinders. Therefore, in view of (4-2), the sequence $(s(j) - s(j - i) - f_n)_{n=1}^\infty$ is a rigidity sequence for T along $(W_n)_{n=1}^\infty$. Hence Lemma 4.2 yields that

$$s(i) = s(i + 1) - s(1) = s(i + 2) - s(2) = \dots = s(r - 1) - s(r - 1 - i).$$

This is equivalent to the following i -periodicity property for σ :

$$\sigma(i + 1) = \sigma(1), \quad \sigma(i + 2) = \sigma(2), \quad \dots, \quad \sigma(r - 1) = \sigma(r - 1 - i).$$

On the other hand, we see that $g_n \in F_{l_n+2} \setminus F_{l_n}$. Since $(C_{l_n+1}, F_{l_n+1}) \sim (r, \sigma)$ and $(C_{l_n+2}, F_{l_n+2}) \sim (r', \sigma')$, it follows that $(C_{l_n+1} + C_{l_n+2}, F_{l_n+2}) \sim (rr', \sigma \diamond \sigma')$ for each $n \geq 0$. Hence replacing C_{l_n+1} in the above argument with $C_{l_n+1} + C_{l_n+2}$ (i.e. passing to the corresponding bounded telescoping), we obtain that the map $\sigma \diamond \sigma'$ is also i -periodic. By the definition of $\sigma \diamond \sigma'$,

$$\sigma \diamond \sigma'(r - i) = \sigma \diamond \sigma'(2r - i) = \cdots = \sigma \diamond \sigma'(r(r' - 1) - i) = \sigma(r - i).$$

Since $\sigma \diamond \sigma'$ is i -periodic, it follows that

$$\sigma \diamond \sigma'(r) = \sigma \diamond \sigma'(2r) = \cdots = \sigma \diamond \sigma'(r(r' - 1))$$

and hence, by the definition of $\sigma \diamond \sigma'$,

$$\sigma'(1) = \sigma'(2) = \cdots = \sigma'(r' - 1).$$

This is equivalent to the fact that C_{l_n+2} is an arithmetic sequence for each $n \geq 0$.

Case B. Let $\delta = 1$ but $i < r - 1$. Then we write $g_n = (f_n - h_n) + (i + 1)h_{l_n}$ for each $n > 0$. Since $i + 1 < r$ and $\lim_{n \rightarrow \infty} \frac{f_n - h_n}{h_{l_n}} = 0 < 1$, we prove the claim in Case B in the very same way as in Case A just by replacing f_n with $f_n - h_n$ and i with $i + 1$ everywhere in the above argument.

Case C. Let $\delta = 1$ and $i = r - 1$. Then $\lim_{n \rightarrow \infty} g_n / h_{l_n+1} = 1$. Hence $2g_n \in F_{l_n+2} \setminus F_{l_n+1}$ for each $n > 0$. The sequence $(2g_n)_{n=1}^\infty$ is also a rigidity sequence for T . Since $\#C_{l_n+2} \geq 4$, it follows that $2g_n / h_{l_n+2} \leq 0.5$ for each $n > 0$. Thus considering $(2g_n)_{n=1}^\infty$ instead of $(g_n)_{n=1}^\infty$ we reduce Case C to Case A or Case B. \square

As a corollary we obtain a characterization of the discrete spectrum of rigid rank-one \mathbb{Z} -actions with bounded parameters. To state this assertion we need a new concept. First, we recall that each odometer $Q = (Q_n)_{n \in \mathbb{Z}}$ can be represented as a minimal rotation on a monothetic compact totally disconnected Abelian group H which is the inverse limit of a sequence of cyclic groups: $\mathbb{Z}/l_1\mathbb{Z} \leftarrow \mathbb{Z}/(l_1 l_2 \mathbb{Z}) \leftarrow \cdots$ associated with a sequence of integers $l_i > 1$ for each $i \in \mathbb{N}$. An element of $q \in Q$ is a sequence $q = (q_i)_{i \in \mathbb{N}}$ such that $0 \leq q_i < l_1 \cdots l_i$ and $q_i \equiv q_{i+1} \pmod{l_1 \cdots l_i}$ for each $i \in \mathbb{N}$. Then $(Q_n q)_i = q_i + n \pmod{l_1 \cdots l_i}$ for each $q \in Q$ and $n \in \mathbb{Z}$. We say that Q is of *bounded type* if the set $\{l_i \mid i \in \mathbb{N}\}$ is finite. It is easy to verify that the bounded type is well defined by Q , i.e. it does not depend on the choice of the “approximating” sequence $(l_i)_{i \in \mathbb{N}}$ ¹⁴.

Proposition 4.5. *Let T be a rigid (C, F) -action of \mathbb{Z} associated with a bounded sequence $(C_n, F_{n-1})_{n \geq 1}$ and let $F_n = \{0, 1, \dots, h_n - 1\}$ for some integers $h_n \geq 1$ for each $n \geq 0$. Then either T is an odometer of bounded type or the group $\Lambda_T \subset \mathbb{T}$ of eigenvalues of T is finite. Moreover, in the latter case, if $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$ for each $n \geq 0$ and $K := \sup_{n \geq 1} \sup_{1 \leq j \leq r_n} \sigma_n(j)$ then the order of each $\lambda \in \Lambda_T$ does not exceed K .*

Proof. Suppose first that there are infinitely many $n \in \mathbb{N}$ for which there exists $m > 2n$ such that the set $C_n + \cdots + C_m$ is an arithmetic sequence but $C_n + \cdots + C_m + C_{m+1}$ is not. We will call such n *good*. Let $\lambda \in \Lambda_T$ and let $\xi : X \rightarrow \mathbb{C}$ be a corresponding

¹⁴In fact, it is a spectral property of Q , i.e. a property of the Koopman operator generated by Q .

non-zero measurable eigenfunction for T , i.e. $\xi \circ T_j = \lambda^j \xi$ almost everywhere on X for each $j \in \mathbb{Z}$. Since T is ergodic, we may assume that $|\xi| = 1$ almost everywhere on X . Fix $\epsilon > 0$. Then we can select a good $n > 0$, an element $f \in F_{n-1}$ and a complex number $z \in \mathbb{T} \subset \mathbb{C}$ such that the set $A := \{x \in X \mid |\xi(x) - z| < \epsilon\}$ is of positive measure and the cylinder $[f]_{n-1}$ is $(1 - \epsilon^2)$ -full of A , i.e.

$$\mu(A \cap [f]_{n-1}) > (1 - \epsilon^2)\mu([f]_{n-1}).$$

By the definition of good numbers, there are integers $a > 0$, $l \in \{0, \dots, \#C_{m+1} - 2\}$ and $v \neq 0$ such that

$$C_n + \dots + C_m = \{0, a, \dots, (p-1)a\} \quad \text{and} \\ C_{m+1} = \{0, pa, 2pa, \dots, lpa, (l+1)pa + v, \dots\},$$

where $p := \#C_n \dots \#C_m$. Since $pa = h_m + \sigma_m(1)$ and $pa + v = h_m + \sigma_m(l+1)$, it follows that $v = \sigma_m(l+1) - \sigma_m(1)$ and hence $|v| \leq K$. Let p' stand for the integer part of $p/3$. Since $[f]_{n-1} = \bigsqcup_{c \in C_n + \dots + C_{m+1}} [f+c]_{m+1}$, it follows that more than $(1 - \epsilon)\#C_n \dots \#C_{m+1}$ cylinders $[f+c]_{m+1}$, $c \in C_n + \dots + C_{m+1}$, are $(1 - \epsilon)$ -full of A . Hence there is $1 < j < p'$ such that the cylinders $[f+ja]_{m+1}$, $[f+ja+p'a]_{m+1}$, $[f+ja+(p-1+lp)a-p'a]_{m+1}$ and $[f+(j-1)a+(l+1)pa+v]_{m+1}$ are $(1 - \epsilon)$ -full of A . Since $[f+ja+p'a]_{m+1} = T_{p'a}[f+ja]_{m+1}$ and

$$[f+(j-1)a+(l+1)pa+v]_{m+1} = T_{p'a+v}[f+ja+(p-1+lp)a-p'a]_{m+1},$$

It follows that there are $x, y \in A$ such that $T_{p'a}x \in A$ and $T_{p'a+v}y \in A$. Hence $|z - \lambda^{p'a}z| < 2\epsilon$ and $|z - \lambda^{p'a+v}z| < 2\epsilon$. This implies that $|1 - \lambda^v| < 4\epsilon$. Since ϵ can be chosen arbitrarily small, $\lambda^v = 1$.

Suppose now that the set of good numbers is finite. Then it follows from Theorem 4.4 that there is $n > 0$ such that the set $C_n + C_{n+1} + \dots$ is an infinite arithmetic sequence. Let $a := \sigma_{n-1}(1)$. Then, as it is well known, T_1 is the integral transformation constructed over the odometer determined by the following approximating sequence of integers: $\#F_{n-1}$, $\#F_{n-1}\#C_n$, $\#F_{n-1}\#C_n\#C_{n+1}$, \dots and under the map that takes constant value a . Hence T_1 is also an odometer. It is determined by the sequence of integers $a\#F_{n-1}$, $a\#F_{n-1}\#C_n$, $a\#F_{n-1}\#C_n\#C_{n+1}$, \dots . Thus we see that T_1 is an odometer of bounded type. \square

4.3. Ergodicity of powers for rank-one transformations. We state here without proof a standard lemma (see [Da3, Lemma 1.2]).

Lemma 4.6. *Let $T = (T_g)_{g \in G}$ be a (C, F) -action of G associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(III) and (1-4). Let $\delta : G \rightarrow (0, 1)$ be a map. If for each $f, f' \in F_n$, there is a subset $A \subset [f]_n$ and $l \in \mathbb{Z}$ such that $\mu(A) \geq \delta(f' - f)\mu([f]_n)$ and $T_{lg}A \subset [f']_n$ then the transformation T_g is ergodic.*

We now state and prove the main result of this subsection.

Theorem 4.7. *Let T be as in Lemma 4.6, $G = \mathbb{Z}$ and $F_n = \{0, 1, \dots, h_n - 1\}$ for some integers $h_n \geq 1$ for each $n \geq 0$.*

- (i) *If T_d is ergodic then for each divisor p of d , there are infinitely many $n > 0$ such that some $c \in C_n$ is not divisible by p .*
- (ii) *If the sequence $(\#C_n)_{n=1}^\infty$ is bounded and for each divisor p of a positive integer d , there are infinitely many n such that p does not divide some $c \in C_n$ then T_d is ergodic.*

Proof. (i) Suppose that there are a divisor p of d and a positive integer N such that for each $n > N$ and each $c \in C_n$, c is divisible by p . Since T_d is ergodic, there is $m > 0$ such that $T_{dm}[0]_N \cap [1]_N \neq \emptyset$. Hence $dm - 1 \in \sum_{n>N} (C_n - C_n)$. It follows that $dm - 1$ is divisible by p , a contradiction.

(ii) Let K be a positive integer such that $\sup_{n>0} \#C_n < K$. We let

$$D := \{j \in \{1, \dots, d-1\} \mid \exists q \in C_n - C_n, q \equiv j \pmod{d} \text{ for infinitely many } n\}.$$

Identifying naturally the set $\{0, 1, \dots, d-1\}$ with the cyclic group $\mathbb{Z}/d\mathbb{Z}$, we consider D as a subset of $\mathbb{Z}/d\mathbb{Z}$. It follows easily from the condition of the theorem that D is non-empty. We claim that D generates $\mathbb{Z}/d\mathbb{Z}$. Indeed, otherwise there is $p > 1$ such that p divides d and $D \subset \{0, p, 2p, \dots, (p_1 - 1)p\} \subset \mathbb{Z}/d\mathbb{Z}$, where $p_1 = d/p$. Moreover, it follows now from the definition of D that there is $N > 0$ such that p divides each element $q \in C_n - C_n$ for every $n > N$. Since $C_n \subset C_n - C_n$, we obtain that p divides every element $c \in C_n$ whenever $n > N$. This contradicts the condition of the theorem.

It follows from the claim that there are a subset $D_0 \subset D$ and non-zero integers α_j , $j \in D_0$, such that

$$(4-4) \quad \sum_{j \in D_0} \alpha_j j \in 1 + d\mathbb{Z}.$$

Now we can find mutually disjoint infinite subsets $\mathcal{L}_j \subset \mathbb{N}$, $j \in D_0$, such that for each $l \in \mathcal{L}_j$, there are $c_l, c'_l \in C_l$ such that $c_l - c'_l - \frac{\alpha_j}{|\alpha_j|} j \in d\mathbb{Z}$.

Suppose that we are given $N > 0$ and $f, f' \in F_N$ such that $m := f - f' > 0$. It follows from (4-4) that $\sum_{j \in D_0} m \alpha_j j \in m + d\mathbb{Z}$. Choose finite subsets $L_j \subset \mathcal{L}_j$ such that $N < \min L_j$ and $\#L_j = m|\alpha_j|$ for each $j \in D_0$. We now set

$$L := \max_{j \in D_0} \max L_j \quad \text{and} \quad A := \left[f + \sum_{j \in D_0} \sum_{l \in L_j} c_l \right]_L.$$

Then

$$\sum_{j \in D_0} \sum_{l \in L_j} (c'_l - c_l) \in \sum_{j \in D_0} \sum_{l \in L_j} \frac{\alpha_j j}{|\alpha_j|} + d\mathbb{Z} = \sum_{j \in D_0} m \alpha_j j + d\mathbb{Z} = m + d\mathbb{Z}.$$

Hence there is $l \in \mathbb{Z}$ such that $dl + m + \sum_{j \in D_0} \sum_{l \in L_j} c_l = \sum_{j \in D_0} \sum_{l \in L_j} c'_l$. This yields

$$T_{dl}A = \left[f - m + \sum_{j \in D_0} \sum_{l \in L_j} c'_l \right]_L \subset [f']_N.$$

Moreover,

$$\mu(A) = \frac{\mu([f]_N)}{\prod_{j \in D_0} \prod_{l \in L_j} \#C_l} \geq \frac{\mu([f]_N)}{\prod_{j \in D_0} K^{m|\alpha_j|}} = \frac{\mu([f]_N)}{(K^{\sum_{j \in D_0} |\alpha_j|})^{f-f'}}.$$

By Lemma 4.6, T_d is ergodic. \square

Corollary 4.8. *Let T be as in Theorem 4.7 and let the sequence $(\#C_n)_{n=1}^\infty$ be bounded. Then T is totally ergodic if and only if for each $d > 1$, there are infinitely many $n > 0$ such that some element c of C_n is not divisible by d .*

We now apply the main results of this section to the Chacon maps.

Example 4.9. (i) Let T be the Chacon map with 2 cuts. Then T is associated with the sequence $(C_n, F_{n-1})_{n=1}^\infty$ such that $F_n = \{0, \dots, h_n - 1\}$, $C_{n+1} = \{0, h_n\}$ and $h_n = 2h_{n-1} + 1 = 2^{n+1} - 1$ for each $n \geq 1$. Hence T is a non-adapted rank-one transformation with bounded parameters. Since $C_{n-1} + C_{n+2} = \{0, h_{n-1}, 2h_{n-1} + 1, 3h_{n-1} + 1\}$ is not an arithmetic sequence for any $n > 0$, it follows from Theorem 4.4 that T is not rigid. Since $h_n = h_{n-1} + 1$, no divisor of h_{n-1} is a divisor of h_n for any $n \in \mathbb{N}$. Then Corollary 4.8 yields that T is totally ergodic.

(ii) Let T be the Chacon map with 3 cuts. Then T is associated with the sequence $(C_n, F_{n-1})_{n=1}^\infty$ such that $F_n = \{0, \dots, h_n - 1\}$, $C_{n+1} = \{0, h_n, 2h_n + 1\}$ and $h_n = 3h_{n-1} + 1 = \frac{3^{n+1} - 1}{2}$ for each $n \geq 1$. Hence T is an adapted rank-one transformation with bounded parameters. Since C_n is not an arithmetic sequence for any $n > 0$, it follows from Theorem 4.4 that T is not rigid. Of course, no divisor of h_{n-1} is a divisor of h_n for any $n \in \mathbb{N}$. Hence T is totally ergodic in view of Corollary 4.8.

5. DISJOINTNESS AND MSJ FOR RANK-ONE TRANSFORMATIONS WITH BOUNDED PARAMETERS

5.1 Joinings and disjointness. We first recall definitions of joinings and disjointness in the sense of Furstenberg (see [dJRu], [Ru] and [Fu] for details). Let $T = (T_n)_{n \in \mathbb{Z}}$ and $R = (R_n)_{n=1}^\infty$ be two ergodic \mathbb{Z} -actions on standard probability spaces (X, μ) and (Y, ν) respectively. A *joining* of T and R is a $(T_1 \times R_1)$ -invariant probability measure on $X \times Y$ whose pullbacks on X and Y are μ and ν respectively. The set of all joinings of T and R is denoted by $J(T, R)$. The subset of ergodic joinings of T and R is denoted by $J^e(T, S)$. We note that $J^e(T, S)$ is the set of extreme points of the convex set $J(T, R)$. If $J^e(T, S) = \{\mu \times \nu\}$ then T and S are called *disjoint*. In the case where $T = S$, we reduce the notation $J^e(T, S)$ to $J_2^e(T)$. Given $\theta \in C(T)$, we define a measure μ_θ on $X \times X$ by setting $\mu_\theta(A \times B) = \mu(A \cap S^{-1}B)$, where A and B are Borel subsets in X . Of course, $\mu_\theta \in J_2^e(T)$ and μ_θ is supported on the graph of θ . It is called a *graph-joining* of T . More generally, each measure preserving isomorphism $\theta : X \rightarrow Y$ intertwining T with S generates a joining from $J^e(T, S)$ that is supported on the graph of θ . If $J_2^e(T) = \{\mu_{T_n} \mid n \in \mathbb{Z}\} \cup \{\mu \times \mu\}$ then T is said to have *minimal self-joinings of order 2* (MSJ_2). It follows, in particular, that $C(T) = \{T_n \mid n \in \mathbb{Z}\}$ and hence T is not rigid if T has MSJ_2 . The property of MSJ of higher orders is defined in a similar way (see [dJRu]).

5.2. Joinings of rank-one transformations with bounded parameters. In this subsection $T = (T_i)_{i \in \mathbb{Z}}$ and $T' = (T'_i)_{i \in \mathbb{Z}}$ are the (C, F) -actions of \mathbb{Z} associated with sequences of parameters $(C_n, F_{n-1})_{n \geq 1}$ and $(C'_n, F'_{n-1})_{n \geq 1}$ respectively. The two sequences satisfy (I)–(III), (1-2) and (1-4). We will also assume that $F_n = \{0, \dots, h_n - 1\}$ and $F'_n = \{0, \dots, h'_n - 1\}$ for some $h_n > 0$ and $h'_n > 0$ for each $n \geq 0$. Let (X, μ) and (X', μ') be the measure spaces of T and T' respectively. Let $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$ and $(C'_{n+1}, F'_{n+1}) \sim (r'_n, \sigma'_n)$ for some maps $\sigma_n : \{1, \dots, r_n\} \rightarrow \mathbb{Z}_+$ and $\sigma'_n : \{1, \dots, r'_n\} \rightarrow \mathbb{Z}_+$ for each $n > 0$. Denote by $s_n :$

$\{0, \dots, r_n - 1\} \rightarrow \mathbb{Z}_+$ and $s'_n : \{0, \dots, r_n - 1\} \rightarrow \mathbb{Z}_+$ the integrals of σ_n and σ'_n respectively, $n \geq 0$.

Proposition 5.1. *Suppose that there exist an infinite subset $D \subset \mathbb{N}$, an integer $r \geq 3$ and two maps $s, s' : \{0, 1, \dots, r - 1\} \rightarrow \mathbb{Z}_+$ such that $h_n = h'_n$, $r_n = r'_n = r$, $s_n = s$ and $s'_n = s'$ whenever $n + 1 \in D$. Then for each $\lambda \in J^e(T, T')$ and every $i_0 \in \{0, 1, \dots, r - 1\}$, there is $i'_0 \in \{0, 1, \dots, r - 1\}$ such that*

$$\lambda = \lambda \circ (T_{s(i+i_0)-s(i_0)} \times T'_{s'(i+i'_0)-s'(i'_0)})$$

for every i such that $-\min(i_0, i'_0) \leq i \leq r - 1 - \max(i_0, i'_0)$.

Proof. Let $\lambda \in J^e(T, T')$. A point $(x, x') \in X \times X'$ is called a *generic point* for λ if

$$(5-1) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n - a_n} \sum_{i=a_n}^{b_n-1} 1_{A \times A'}(T_i x, T'_i x') = \lambda(A \times A')$$

for all cylinders $A \subset X$ and $A' \subset X'$ whenever $b_n > 0$, $a_n \leq 0$ and $b_n - a_n \rightarrow +\infty$ as $n \rightarrow \infty$. By the individual ergodic theorem, λ -a.e. point of $X \times X'$ is generic. Fix $m > 0$ such that $x = (f_m, c_{m+1}, c_{m+2}, \dots) \in X_m$ and $x' = (f'_m, c'_{m+1}, c'_{m+2}, \dots) \in X'_m$. Fix $i_0 \in \{0, \dots, r - 1\}$. Since λ projects onto μ and the measure $\mu \upharpoonright X_m$ is proportional to the infinite product of the equidistributions on F_m and on C_j for all $j > m$, we may assume without loss of generality (passing to a subsequence of D if necessary) that $3h_n/4 \geq f_n \geq h_n/4$ and $c_{n+1} = s(i_0) + i_0 h_n$ whenever $n + 1 \in D$. Moreover, passing to a further subsequence in D , we may also assume that there is $i'_0 \in \{0, \dots, r - 1\}$ such that $c'_{n+1} = s'(i'_0) + i'_0 h_n$ if $n + 1 \in D$ and there exist

$$(5-2) \quad \lim_{D \ni n+1 \rightarrow \infty} f_n/h_n = \delta \in [0.25, 0.75] \quad \text{and} \quad \lim_{D \ni n+1 \rightarrow \infty} f'_n/h_n = \delta'.$$

Of course, $0 \leq \delta' \leq 1$. Let A and A' be two cylinders in X and X' respectively. We represent them as $A = [A_n]_n$ and $A' = [A'_n]_n$ with $A_n \cup A'_n \subset F_n$ for all sufficiently large n . We will also assume that $A_n \pm \max_{0 < i < r} s(i) \subset F_n$ and $A'_n \pm \max_{0 < i < r} s'(i) \subset F_n$ if $n + 1 \in D$. We now let

$$a_n := \begin{cases} -f_n & \text{if } f'_n \geq f_n \\ -f'_n & \text{if } f'_n \leq f_n \end{cases} \quad \text{and} \quad b_n := \begin{cases} h_n - f'_n & \text{if } f'_n \geq f_n \\ h_n - f_n & \text{if } f'_n \leq f_n. \end{cases}$$

Then $a_n \leq 0$, $b_n > 0$ and $b_n - a_n \rightarrow +\infty$ as $n \rightarrow \infty$. If $i \in [a_n, b_n)$ then the point

$$T_i x = T_i(f_n, c_{n+1}, \dots) = (i + f_n, c_{n+1}, \dots)$$

belongs to A if and only if $i + f_n \in A_n$. In a similar way, $T'_i x' \in A'$ if and only if $i + f'_n \in A'_n$. The following four cases are possible: $f'_n \geq f_n$ and $f'_{n+1} \geq f_{n+1}$, $f'_n \geq f_n$ and $f'_{n+1} \leq f_{n+1}$, $f'_n \leq f_n$ and $f'_{n+1} \geq f_{n+1}$, and $f'_n \leq f_n$ and $f'_{n+1} \leq f_{n+1}$. We consider only the first case because the other ones are similar. It follows from (5-1) that

$$(5-3) \quad \begin{aligned} \lambda(A \times A') &= \frac{\#((A_n - f_n) \cap (A'_n - f'_n) \cap [a_n, b_n))}{h_n + f_n - f'_n} + \bar{o}(1) \\ &= \frac{\#((A_n - f_n) \cap (A'_n - f'_n))}{h_n + f_n - f'_n} + \bar{o}(1) \end{aligned}$$

eventually in n . As usual $\bar{o}(1)$ denotes a sequence going to 0 as $n \rightarrow \infty$. Take $n+1 \in D$. Since $f_{n+1} = f_n + c_{n+1}$ and $f'_{n+1} = f'_n + c'_{n+1}$, it follows that

$$A_{n+1} - f_{n+1} = A_n - f_n + C_{n+1} - c_{n+1} = \bigsqcup_{i=-i_0}^{r-1-i_0} (A_n - s(i_0) + s(i+i_0) - f_n + ih_n)$$

and, in a similar way,

$$A'_{n+1} - f'_{n+1} = \bigsqcup_{i=-i'_0}^{r-1-i'_0} (A'_n - s'(i'_0) + s'(i+i'_0) - f'_n + ih_n).$$

We let $\hat{i}_0 := \min(i_0, i'_0)$, $\tilde{i}_0 := \max(i_0, i'_0)$, $B_{n,i} := A_n - s(i_0) + s(i+i_0)$ and $B'_{n,i} := A'_n - s'(i'_0) + s'(i+i'_0)$. Then $B_{n,i} \cup B'_{n,i} \subset F_n$ and

$$(A_{n+1} - f_{n+1}) \cap (A'_{n+1} - f'_{n+1}) \supset \bigsqcup_{i=-\hat{i}_0}^{r-1-\tilde{i}_0} ((B_{n,i} - f_n) \cap (B'_{n,i} - f'_n) + ih_n).$$

Now (5-3) yields that

$$\begin{aligned} \lambda(A \times A') &\geq \frac{\sum_{i=-\hat{i}_0}^{r-1-\tilde{i}_0} \#((B_{n,i} - f_n) \cap (B'_{n,i} - f'_n))}{h_{n+1} + f_{n+1} - f'_{n+1}} + \bar{o}(1) \\ &= \frac{(h_n + f_n - f'_n) \sum_{i=-\hat{i}_0}^{r-1-\tilde{i}_0} \lambda([B_{n,i}]_n \times [B'_{n,i}]_n)}{h_{n+1} + f_{n+1} - f'_{n+1}} + \bar{o}(1) \end{aligned}$$

as $D \ni n+1 \rightarrow \infty$. Applying (5-2) we obtain that

$$\lambda(A \times A') + \bar{o}(1) \geq \frac{1 + \delta - \delta'}{r + \delta - i_0} \sum_{i=-\hat{i}_0}^{r-1-\tilde{i}_0} \lambda \circ (T_{s(i+i_0)-s(i_0)} \times T'_{s'(i+i'_0)-s'(i'_0)}) ([A_n]_n \times [A'_n]_n)$$

as $D \ni n+1 \rightarrow \infty$. It follows that

$$\lambda \gg \sum_{i=-\hat{i}_0}^{r-1-\tilde{i}_0} \lambda \circ (T_{s(i+i_0)-s(i_0)} \times T'_{s'(i+i'_0)-s'(i'_0)}).$$

Of course, $\lambda \circ (T_{s(i+i_0)-s(i_0)} \times T'_{s'(i+i'_0)-s'(i'_0)}) \in J^e(T, T')$ for each $i = -\hat{i}_0, -\hat{i}_0 + 1, \dots, r-1-\tilde{i}_0$. By the extremal property of the ergodic joinings,

$$\lambda \circ (T_{s(i+i_0)-s(i_0)} \times S_{s'(i+i'_0)-s'(i'_0)}) = \lambda$$

for each $i = -\hat{i}_0, \hat{i}_0 + 1, \dots, r-1-\tilde{i}_0$. \square

We recall a classical lemma by Furstenberg.

Lemma 5.2. *Given two standard probability spaces (X, μ) and (Y, ν) and a measure λ on $X \times Y$ whose pullbacks onto X and Y are μ and ν respectively, if there is an ergodic ν -preserving invertible transformation R of Y such that $\lambda \circ (Id \times R) = \lambda$ then $\lambda = \mu \times \nu$.*

Now we can prove Ryzhikov's theorem from [Ry].

Theorem 5.3. *Let T be a rank-one \mathbb{Z} -action with bounded parameters. Suppose that T is not rigid and that T is totally ergodic. Then T has MSJ_2 and hence MSJ .*

Proof. Without loss of generality (due to Lemma 1.5) we may assume without loss of generality that T is the (C, F) -action associated with bounded parameters $(C_n, F_{n-1})_{n \geq 1}$ and $F_n = \{0, \dots, h_n - 1\}$ for some $h_n > 0$. Let r_n, σ_n and s_n be the same objects as in the beginning of §5.2. Passing, if necessary, to a bounded telescoping, we may assume that $r_n \geq 4$ for all $n \geq 1$. Take a joining $\lambda \in J_2^e(T)$ which is not a graph-joining. Since the cutting-and-stacking parameters are bounded, there is $r \geq 2$, an infinite subset $D \subset \mathbb{N}$ and a map $\sigma : \{1, \dots, r\} \rightarrow \mathbb{Z}_+$ such that $\sigma_n = \sigma$ whenever $n + 1 \in D$. Let s be the integral of σ . Then $s_n = s$ whenever $n + 1 \in D$. We will use the notation from the proof of Proposition 5.1. By that proposition, if $(x, x') = ((f_m, c_{m+1}, \dots), (f'_m, c'_{m+1}, \dots))$ is a generic point for λ and λ is not a graph-joining then there exists a pair (i_0, i'_0) such that $i_0 \neq i'_0$ such that

$$(5-4) \quad \lambda = \lambda \circ (T_{s(i+i_0)-s(i_0)} \times T_{s(i+i'_0)-s(i'_0)})$$

for every $i \in \{-\widehat{i_0}, \dots, r-1-\widetilde{i_0}\}$. Indeed, otherwise, i.e. in the situation that $i_0 = i'_0$ for each $i_0 \in \{0, \dots, r-1\}$ for every choice of D , due to the boundedness condition, we would have that $c_n = c'_n$ eventually in n and hence x' belongs to the T -orbit of x . Then a standard reasoning implies that λ is a graph-joining.

Next, we may not to consider generic points (x, x') which are “extreme”, i.e. such points that $\{c_{n+1}, c'_{n+1}\} = \{0, s_n(r_n - 1) + (r_n - 1)h_n\}$ eventually in n , because the λ -measure of the set of all extreme points is 0¹⁵. Therefore, choosing D in an appropriate way, we may assume that $\{c_{n+1}, c'_{n+1}\} \neq \{0, s(r-1) + (r-1)h_n\}$ for each sufficiently large $n + 1 \in D$. This means that

$$(5-5) \quad \{i_0, i'_0\} \neq \{0, r-1\}.$$

Since T is totally ergodic and (5-4) holds, we deduce from Lemma 5.2 that $\lambda \neq \mu \times \mu$ only if $s(i+i_0) - s(i_0) = s(i+i'_0) - s(i'_0)$ for all $i \in \{-\widehat{i_0}, \dots, r-1-\widetilde{i_0}\}$. The latter condition (in view of (5-5)) is equivalent to the following one: $\sigma(i) = \sigma(i + |i'_0 - i_0|)$ for each $i = 1, \dots, r-1 - |i'_0 - i_0|$, i.e. σ is $|i'_0 - i_0|$ -periodic. Arguing as in the proof of Theorem 4.4 we obtain that T is rigid, a contradiction. Hence $\lambda = \mu \times \mu$ and thus T has MSJ_2 .

Since T is a rank-one action with bounded parameters, it is a routine to show that T is *partially rigid*, i.e. there exist a sequence $n_k \rightarrow \infty$ and a real parameter $\eta > 0$ such that $\liminf_{k \rightarrow \infty} \mu(T_{n_k} A \cap A) \geq \eta \mu(A)$ for some for each Borel subset $A \subset X$. Therefore every factor¹⁶ of T is also partially rigid (with the same parameter η)

¹⁵This follows from the fact that the projection of the set of extreme points on each coordinate of $X \times X$ is of μ -measure 0.

¹⁶A *factor* of T is an invariant sub- σ -algebra of Borel subsets in X or, more rigorously, the restriction of (T, μ) to this sub- σ -algebra.

and hence T has no factor with Lebesgue spectrum. Since T has MSJ_2 , it follows now from [GIHoRu, Theorem 4] that T has MSJ of all orders. \square

It follows from this theorem and Example 4.9 that the Chacon 3-cuts map and the Chacon 2-cuts maps have MSJ.

The next claim follows from Theorem 5.3 and the properties of transformations with MSJ (see, for instance, [dJRu, Corollary 6.5]).

Corollary 5.4. *If $n, m > 0$ and $n \neq m$ then T_n and T_m are disjoint.*

We now deduce one more corollary from Proposition 5.1. We first note that if T and T' are commensurate and the parameters $(C_n, F_{n-1})_{n=1}^\infty$ and $(C'_n, F'_{n-1})_{n=1}^\infty$ are bounded then $r_n = r'_n$ (i.e. $\#C_n = \#C'_n$) eventually.

Corollary 5.5. *Let T and T' be two commensurate (C, F) -actions of \mathbb{Z} associated with bounded parameters $(C_n, F_{n-1})_{n=1}^\infty$ and $(C'_n, F'_{n-1})_{n=1}^\infty$ respectively. Let T or T' not be rigid.*

- (i) *Then T and T' are isomorphic if and only if $s_n = s'_n$ eventually.*
- (ii) *If $s_n \neq s'_n$ for infinitely many n and for each $n > 0$, either T_n or T'_n is ergodic then T and T' are disjoint.*

We preface the proof of the corollary with a simple auxiliary lemma.

Lemma 5.6. *Given positive integers r and q , let $\sigma, \omega : \{1, \dots, r\} \rightarrow \mathbb{Z}_+$ and $\alpha, \beta : \{1, \dots, q\} \rightarrow \mathbb{Z}_+$ be four maps. Suppose that there is $i_0 > 0$ such that $i_0 \leq rq/2$, r does not divide i_0 , and $\sigma \diamond \alpha(i) = \omega \diamond \beta(i + i_0)$ for each $i \in \{1, \dots, rq - i_0 - 1\}$. Then there is $p \leq q/2$ such that $\alpha(1) = \alpha(2) = \dots = \alpha(q - p)$ and $\beta(p) = \beta(p + 1) = \dots = \beta(q)$.*

Proof. There is $z \in \{1, \dots, r - 1\}$ such that $i_0 = j_0 r + z$ for some non-negative integer $j_0 < q/2$. Then for each $j \in \{1, \dots, q - j_0 - 1\}$,

$$\sigma(r) + \alpha(j) = \sigma \diamond \alpha(jr) = \omega \diamond \beta(jr + i_0) = \omega \diamond \beta((j + j_0)r + z) = \omega(z).$$

Hence $\alpha(1) = \alpha(2) = \dots = \alpha(q - j_0 - 1)$. In a similar way, for each $j \in \{j_0 + 1, \dots, q\}$,

$$\omega(r) + \beta(j) = \omega \diamond \beta(jr) = \sigma \diamond \alpha(jr - i_0) = \omega \diamond \beta((j - j_0 - 1)r + r - z) = \omega(r - z).$$

Hence $\beta(q) = \beta(q - 1) = \dots = \beta(j_0 + 1)$. \square

Proof of Corollary 5.5. (i) The “if” part is trivial. We now prove the “only if” part. Thus suppose that T and T' are isomorphic. Let T be rigid. The case where T' is rigid is considered in a similar way. Let $\lambda \in J^e(T, S)$ denote the graph joining generated by this isomorphism. Since the parameters $(C_n, F_{n-1})_{n=1}^\infty$ and $(C'_n, F'_{n-1})_{n=1}^\infty$ are bounded, there exist $r > 0$, an infinite subset $D \subset \mathbb{N}$ and maps $s, s' : \{0, \dots, r - 1\} \rightarrow \mathbb{Z}_+$ such that $s_n = s$ and $s'_n = s'$ whenever $n + 1 \in D$. Let i_0 stand for the integer part of $(r - 1)/2$. By Proposition 5.1, there is $i'_0 \in \{0, 1, \dots, r - 1\}$ such that

$$(5-6) \quad \lambda = \lambda \circ (T_{s(i+i_0)-s(i_0)-s'(i+i'_0)+s'(i'_0)} \times \text{Id})$$

for every i such that $-\min(i_0, i'_0) \leq i \leq r - 1 - \max(i_0, i'_0)$. Since the conditional measures of λ corresponding to the projection $X \times X' \ni (x, x') \mapsto x' \in X'$ are

delta-measures almost everywhere on X' and T is free, it follows from (5-6) that $T_{s(i+i_0)-s(i_0)-s'(i+i'_0)+s'(i'_0)} = \text{Id}$ and hence $s(i+i_0) - s(i_0) - s'(i+i'_0) + s'(i'_0) = 0$ for every i such that $-\min(i_0, i'_0) \leq i \leq r-1 - \max(i_0, i'_0)$. This is equivalent to

$$(5-7) \quad \sigma(i) = \sigma'(i + |i'_0 - i_0|) \text{ or } \sigma'(i) = \sigma(i + |i'_0 - i_0|)$$

(depending on the sing of $i_0 - i'_0$) for each $i = 1, \dots, r-1 - |i'_0 - i_0|$. We note that $|i'_0 - i_0| \leq r/2$. Passing to a subsequence in D we may assume also that the sequences $(\sigma_n)_{n \in D}$, $(\sigma_{n+1})_{n \in D}$, $(\sigma'_n)_{n \in D}$ and $(\sigma'_{n+1})_{n \in D}$ are all constant, i.e. there are $p, q > 1$ and maps $\alpha, \alpha' : \{1, \dots, q\} \rightarrow \mathbb{Z}_+$ and $\beta, \beta' : \{1, \dots, q\} \rightarrow \mathbb{Z}_+$ such that $\sigma_n = \alpha$, $\sigma_{n+1} = \beta$, $\sigma'_n = \alpha'$ and $\sigma'_{n+1} = \beta'$ for each $n \in D$. Thus we now obtain that $(C_n, F_n) \sim (r, \sigma)$, $(C_{n+1}, F_{n+1}) \sim (q, \alpha)$, $(C_{n+2}, F_{n+2}) \sim (p, \beta)$, $(C'_n, F'_n) \sim (r, \sigma')$, $(C'_{n+1}, F'_{n+1}) \sim (q, \alpha')$ and $(C'_{n+2}, F'_{n+2}) \sim (p, \beta')$ for each $n \in D$. Since $(C_n + C_{n+1} + C_{n+2}, F_{n+2}) \sim (pqr, \sigma \diamond (\alpha \diamond \beta))$ and $(C'_n + C'_{n+1} + C'_{n+2}, F'_{n+2}) \sim (pqr, \sigma' \diamond (\alpha' \diamond \beta'))$, we pass to a corresponding bounded telescoping to deduce from (5-7) that

$$\begin{aligned} \sigma \diamond (\alpha \diamond \beta)(i) &= \sigma' \diamond (\alpha' \diamond \beta')(i + j_0) \quad \text{or} \\ \sigma' \diamond (\alpha' \diamond \beta')(i) &= \sigma \diamond (\alpha \diamond \beta)(i + j_0) \end{aligned}$$

for some positive integer $j_0 \leq pqr/2$ and each $i = 1, \dots, pqr - j_0$. It follows from Lemma 5.6 that $\alpha \diamond \beta(1) = \dots = \alpha \diamond \beta(l_0)$ or $\alpha \diamond \beta(pq) = \dots = \alpha \diamond \beta(pq - l_0)$ for some positive integer $l_0 \leq pq/2$. This implies, in turn, that $\alpha(1) = \dots = \alpha(q-1)$ is constant. Hence C_{n+1} is an arithmetic sequence for each $n \in D$. It follows now from Theorem 4.4 (or, more precisely, from the claim in the beginning of the proof of Theorem 4.4) that T is rigid, a contradiction.

(ii) is proved in a similar way. We only note that (5-6), the ergodicity condition in the statement of (ii) and Lemma 5.2 imply that $\lambda = \mu \times \mu'$, i.e. that T and S are disjoint whenever $s(i+i_0) - s(i_0) - s'(i+i'_0) + s'(i'_0) \neq 0$ for some i such that $-\min(i_0, i'_0) \leq i \leq r-1 - \max(i_0, i'_0)$. Thus, if T and S are not disjoint then we obtain as in (i) that (5-7) holds. As we have already shown in the proof of (i), the condition (5-7) (because it holds for every bounded telescoping of the original (C, F) -parameters of T and T') implies that T is rigid, a contradiction. \square

We consider an application of Corollary 5.5 to the inverse problem.

Corollary 5.7. *Let T be a (C, F) -action of \mathbb{Z} with bounded parameters. If T is not rigid then T and T^{-1} are isomorphic if and only if $C_n^* = C_n$ eventually¹⁷. If, moreover, T is totally ergodic and T is not isomorphic to T^{-1} then T and T^{-1} are disjoint.*

To prove the corollary we need a measure theoretical analogue of Lemma 2.1.

Lemma 5.8. *Let $T = (T_g)_{g \in G}$ and $T' = (T'_g)_{g \in G}$ be two ergodic measure preserving free G -actions on σ -finite standard measure spaces (X, μ) and (X', μ') respectively. Let A be a Borel subset in X and let A' be a Borel subset in X' such that $\mu(A) = \mu'(A') > 0$. If there is a measure preserving Borel bijection $\theta : A \rightarrow A'$ such that $\theta T_g x = T'_g \theta x$ for each $x \in A$ and $g \in G$ such that $T_g x \in A$ then T and T' are (measure theoretically) conjugate.*

¹⁷For the definition of C_n^* (and F_n^* used in the proof of Corollary 5.7) see Lemma 2.8.

Proof. Since T is ergodic, there is a countable partition of $X \setminus A$ into subsets A_g , $g \in G \setminus \{1\}$, such that $T_g^{-1}A_g \subset A$ for each $g \in G$, $g \neq 1$. We now set

$$\tilde{\theta}x = \begin{cases} \theta x, & \text{if } x \in A \\ T'_g \theta T_g^{-1}x, & \text{if } x \in A_g. \end{cases}$$

It is routine to verify that $\tilde{\theta}$ is Borel, one-to-one and onto (mod 0). Of course, θ is measure preserving. It is obvious that $\tilde{\theta}$ intertwines T with T' . \square

Proof of Corollary 5.7. We note that T^{-1} is a (C, F) -action associated with the sequence $(C_n^*, F_{n-1}^*)_{n \geq 1}$. It is easy to verify that the sequence $(C_n^*, F_{n-1}^*)_{n \geq 1}$ satisfies the conditions (I)–(III) and (1-4). It is bounded. The (C, F) -action of \mathbb{Z} associated with $(C_n^*, F_{n-1}^*)_{n \geq 1}$ is isomorphic to T^{-1} by Lemma 5.8. It remains to apply Corollary 5.5. \square

Corollary 5.7 yields immediately that the Chacon map with 3 cuts and the Chacon map with 2 cuts are (measure theoretically) isomorphic to their inverses.

5.3. Light mixing and bounded rank-one constructions. Our purpose here is to show that the class of all rank-one transformations with bounded parameters is strictly bigger than the subclass of adapted rank-one transformations with bounded parameters. For that we will use the property of light mixing. Let $S = (S_n)_{n \in \mathbb{Z}}$ be a measure preserving \mathbb{Z} -action on a standard probability space (Y, ν) . We recall that S is called *lightly mixing* (see, for instance, [Si]) if for all subsets $A, B \subset Y$ of positive measure,

$$\liminf_{n \rightarrow \infty} \nu(S_n A \cap B) > 0.$$

By [FrKi], the Chacon map with 2 cuts is lightly mixing. We now prove the following theorem.

Theorem 5.9. *Let $T = (T_n)_{n \in \mathbb{Z}}$ be an adapted rank-one action of \mathbb{Z} and let $T_1 \sim (r_n, \sigma_n)_{n \geq 1}$ for some sequence of integers r_n and maps $\sigma_n : \{1, \dots, r_n\} \rightarrow \mathbb{Z}_+$. If there is $K > 0$ such that $\sup_{n \geq 1} \max_{1 \leq i \leq r_n} \sigma_n(i) \leq K$ then there are a sequence $n_m \rightarrow +\infty$ and a polynomial $P(Z) = \nu_0 + \nu_1 Z + \dots + \nu_K Z^K$ with real coefficients such that $\min_{0 \leq i \leq K} \nu_i \geq 0$, $\sum_{0 \leq i \leq K} \nu_i = 1$ and*

$$(5-8) \quad T_{-h_{n_m}} \rightarrow P(T_1)$$

*in the weak operator topology*¹⁸. It follows that T is not lightly mixing.

Proof. Represent T as a (C, F) -action associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ such that $F_n = \{0, 1, \dots, h_n - 1\}$ and $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$ for each $n \geq 0$. As above, we will denote by s_n the integral of σ_n . Since T is adapted, i.e. $\sigma_n(r_n) = 0$ for each n , it follows that

$$\sup_{n, n > 0} \max_{1 \leq i \leq r_n \dots r_{n+m}} (\sigma_n \diamond \dots \diamond \sigma_{n+m})(i) \leq K.$$

Therefore, passing, if necessary, to a telescoping we may assume without loss of generality that $r_n \rightarrow \infty$ while the upper bound on the number of consecutive

¹⁸We consider T_i in (5-8) as the unitary Koopman operators in $L^2(X, \mu)$ generated by the transformation T_i , $i \in \mathbb{Z}$.

spacers does not increase. Let A and B be cylinders in X . Then $A = [A_n]_n$ and $B = [B_n]_n$ for some subsets A_n and B_n in \mathbb{Z} such that $A_n \cup B_n \subset \{0, \dots, h_n - 1 - K\}$ for each sufficiently large n . Then

$$\begin{aligned}
\mu(T_{-h_n}A \cap B) &= \mu(T_{-h_n}[A_n + C_{n+1}]_{n+1} \cap B) \\
&= \mu\left(\bigsqcup_{1 \leq i < r_n} [\sigma_n(i) + A_n + s_n(i-1) + (i-1)h_n]_{n+1} \cap [B_n]_n\right) \pm \frac{1}{r_n} \\
&= \mu\left(\bigsqcup_{1 \leq i < r_n} [(\sigma_n(i) + A_n) \cap B_n + s_n(i-1) + (i-1)h_n]_{n+1}\right) \pm \frac{1}{r_n} \\
&= \frac{1}{r_n} \mu\left(\bigsqcup_{1 \leq i < r_n} [(\sigma_n(i) + A_n) \cap B_n]_n\right) \pm \frac{1}{r_n} \\
&= \frac{1}{r_n} \mu\left(\bigsqcup_{1 \leq i \leq r_n} [\sigma_n(i) + A_n]_n \cap [B_n]_n\right) \pm \frac{2}{r_n} \\
&= \frac{1}{r_n} \mu\left(\bigsqcup_{1 \leq i \leq r_n} T_{\sigma_n(i)}A \cap B\right) \pm \frac{2}{r_n}.
\end{aligned}$$

Denote by ϑ_n the image of the equidistribution on $\{1, \dots, r_n\}$ under σ_n . Then ϑ_n is a distribution on $\{0, 1, \dots, K\}$ and

$$(5-9) \quad \mu(T_{-h_n}A \cap B) = \sum_{i=1}^K \vartheta_n(i) \mu(T_i A \cap B) \pm \frac{2}{r_n}.$$

Let ν stand for a limit point of the sequence $(\vartheta_n)_{n=1}^\infty$. Passing to the limit in (5-9) along a corresponding subsequence we obtain (5-8).

To prove the second claim of the proposition, select subsets A and B of positive measure in such a way that $B \cap \bigcup_{i=0}^K T_i A = \emptyset$. Then, in view of (5-9), we have that $\mu(T^{-h_{n_m}}A \cap B) \rightarrow 0$ as $m \rightarrow \infty$. Hence T is not lightly mixing. \square

5.4. Quadchotomy theorem. We conclude this section with a quadchotomy theorem on “structure” of rank-one transformations with bounded parameters. It refines the trichotomy theorem from [EALedR].

Theorem 5.10. *Let T be a (C, F) -action of \mathbb{Z} associated with a bounded sequence $(C_n, F_{n-1})_{n \geq 1}$ and let $F_n = \{0, 1, \dots, h_n - 1\}$ for some integers $h_n \geq 1$ for each $n \geq 0$. Let $(C_{n+1}, F_{n+1}) \sim (r_n, \sigma_n)$ for each $n \geq 0$ and $K := \sup_{n \geq 1} \sup_{1 \leq j \leq r_n} \sigma_n(j)$. Then one of the following four properties holds:*

- (i) T has MSJ (in particular, T is weakly mixing and $C(T) = \{T_n \mid n \in \mathbb{Z}\}$).
- (ii) T is non-rigid, the group $\Lambda_T \subset \mathbb{T}$ of eigenvalues of T is nontrivial but finite and the order of each $\lambda \in \Lambda_T$ does not exceed K . For each $\rho \in J_2^e(T)$ which is neither a graph-joining nor $\mu \times \mu$, there is $\lambda \in \Lambda_T \setminus \{1\}$ and $n > 0$ such that $\lambda^n = 1$ and $\frac{1}{n} \sum_{i=0}^{n-1} \rho \circ (I \times T_i) = \mu \times \mu$.
- (iii) T is rigid, the group $\Lambda_T \subset \mathbb{T}$ of eigenvalues of T is finite and the order of each $\lambda \in \Lambda_T$ does not exceed K .
- (iv) T is an odometer of bounded type.

Proof. Consider two cases: T is rigid and T is not rigid. If T is rigid then we apply Proposition 4.5 and obtain either (iii) or (iv). If T is not rigid but T is totally ergodic, we apply Theorem 5.3 and obtain (i). It remains to show that if T is non-rigid but not totally ergodic then (iii) holds. Since T is non-rigid, it follows from Theorem 4.4 that there is $M > 0$ such that for each $n > 0$, the sum $C_n + \dots + C_{n+M}$ is not an arithmetic sequence. Let $\lambda \in \Lambda_T$ and let $\xi : X \rightarrow \mathbb{C}$ be a corresponding non-zero measurable eigenfunction for T such that $|\xi| = 1$ almost everywhere on X . Fix $\epsilon > 0$. Then we can select $n > 0$, an element $f \in F_{n-1}$ and a complex number $z \in \mathbb{T} \subset \mathbb{C}$ such that the set $A := \{x \in X \mid |\xi(x) - z| < \epsilon\}$ is of positive measure and the cylinder $[f]_{n-1}$ is $(1 - \epsilon/r^M)$ -full of A . Let $m \geq 0$ be the smallest integer such that the set $C := C_n + \dots + C_{n+m}$ is not an arithmetic sequence. Of course, $m \leq M$. It follows that for each $c \in C$, the cylinder $[f+c]_{n+m}$ is $(1 - \epsilon)$ -full of A . We note that $C = \{ih_{n-1} + s(i) \mid 0 \leq i < \#C\}$ for some map $s : \{0, 1, \dots, \#C - 1\} \rightarrow \mathbb{Z}_+$ and $0 \leq s(i+1) - s(i) \leq \sigma_{n+m-1}(i+1) \leq K$ for each $i = 0, 1, \dots, \#C - 2$. Since for each $i \in \{0, \dots, \#C - 2\}$, we have that $T_{h_{n-1}+s(i+1)-s(i)}[f+ih_{n-1}+s(i)]_{n+m} = [f+(i+1)h_{n-1}+s(i+1)]_{n+m}$, there exists $x_i \in A$ such that $T_{h_{n-1}+s(i+1)-s(i)}x_i \in A$. This yields that

$$\max_{0 \leq i \leq \#C-2} |\lambda^{h_{n-1}+s(i+1)-s(i)} - 1| \leq 2\epsilon.$$

In turn, this inequality implies that $\lambda^k = 1$ for some $k \in \{1, \dots, K\}$. Indeed, otherwise we would obtain that $s(i+1) - s(i) = s(j+1) - s(j)$ for all $0 \leq i, j \leq \#C - 1$ and hence C is an arithmetic sequence, a contradiction. Thus the first statement of (iii) is proved. If $\rho \in J_2^e(T)$ is not a graph-joining, it follows from the proof of Theorem 5.3 that $\rho(I \times T_n) = \rho$ for some $n > 0$. If T_n is ergodic then $\rho = \mu \times \mu$ by Lemma 5.2. Otherwise, there is $\lambda \in \Lambda_T \setminus \{1\}$ such that $\lambda^n = 1$. Then $\omega := \frac{1}{n} \sum_{i=0}^{n-1} \rho \circ (I \times T_i) \in J_2(T)$ and $\omega(I \times T_1) = \omega$. Hence $\omega = \mu \times \mu$ by Lemma 5.2. \square

APPENDIX. SYMBOLIC REPRESENTATION FOR RANK-ONE TRANSFORMATIONS

Let T be a (C, F) -action of \mathbb{Z} associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (I)–(III) and such that $F_n = \{0, 1, \dots, h_n - 1\}$ for all n and some sequence $(h_n)_{n=1}^\infty$. The following definition is equivalent to [AdFePe, Definition 5.1].

Definition A.1. For $k \in \mathbb{Z}_+$, a rank-one (C, F) -action $T = (T_i)_{i \in \mathbb{Z}}$ is *essentially k -expansive* if the partition \mathcal{P}_k of X , given by $\mathcal{P}_k := \{[0]_k, X \setminus [0]_k\}$, generates the entire Borel σ -algebra under the action of T .

For $k \in \mathbb{Z}_+$ and $x \in X$, we set

$$\mathcal{P}_k(x) := \begin{cases} 0, & \text{if } x \in [0]_k \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that the map

$$\phi^{(k)} : X \ni x \mapsto \phi^{(k)}(x) = (\mathcal{P}_k(T_i x))_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$$

is continuous. It intertwines T with the shiftwise \mathbb{Z} -action S on $\{0, 1\}^{\mathbb{Z}}$. Then T is essentially k -expansive if and only if $\phi^{(k)}$ is an isomorphism of (X, μ, T) onto

$(\{0, 1\}^{\mathbb{Z}}, \mu \circ (\phi^{(k)})^{-1}, S)$. Thus $\phi^{(k)}$ provides a *constructive symbolic model* of T (see a discussion about symbolic models in [AdFePe] and [Fe2]). Of course, if T is 0-expansive then T is k -expansive for each $k > 0$. By [Da6, Theorem 2.8, Remark 2.10], each rank-one transformation is (measure theoretically) isomorphic to a rank-one transformation $Q \sim (r'_n, \sigma'_n)_{n=1}^{\infty}$ with $\sigma'_n(r'_n) > 0$ for infinitely many n . If Q is, in addition, probability preserving then—as was explained in [AdFePe]— Q is essentially 0-expansive in view of [Ka, Appendix]. Below (see Theorem A.3) we slightly modify the proof of [Da6, Theorem 2.8] to obtain a finer condition on $(r'_n, \sigma'_n)_{n=1}^{\infty}$ which yields immediately the 0-expansiveness of Q . We note that Theorem A.3 was proved originally in [AdFePe] in a different way using *adic* representations of rank-one systems. Our approach based on the (C, F) -construction leads to a shorter proof.

The following proposition is a slight modification of [AdFePe, Proposition 5.2]. We give a full proof of it to make the present paper more self-contained.

Proposition A.2. *Let $T_1 \sim (r_n, \sigma_n)_{n=1}^{\infty}$ such that (1-5) and (1-6) hold. If $\sigma_n(r_n) > \max\{\sigma_n(1), \dots, \sigma_n(r_n - 1)\}$ for each $n > 0$ then T is essentially 0-expansive.*

Proof. Let \mathfrak{B}_0 stand for the smallest T -invariant sub- σ -algebra containing $[0]_0$. Of course, if $[0]_n \in \mathfrak{B}_0$ for each $n \geq 0$ then \mathfrak{B}_0 coincides with the entire Borel σ -algebra. We prove these inclusions by induction. The inclusion $[0]_0 \in \mathfrak{B}_0$ is by definition. Suppose that $[0]_n \in \mathfrak{B}_0$ for some $n \geq 0$. It follows from the condition of the proposition that

$$[0]_{n+1} = \{x \in [0]_n \mid T_{h_n + \sigma_n(i)}x \notin [0]_n \text{ for each } i = 1, \dots, r_n - 1\},$$

and hence $[0]_{n+1} = [0]_n \cap \bigcap_{i=1}^{r_n-1} T_{-h_n - \sigma_n(i)}(X \setminus [0]_n) \in \mathfrak{B}_0$. \square

We now state and prove the main result of Appendix.

Theorem A.3. *Every rank-one finite measure preserving \mathbb{Z} -action T is (measure theoretically) isomorphic to a (C, F) -action which is essentially 0-expansive.*

Proof. Let $T \sim (r_n, \sigma_n)_{n=1}^{\infty}$ such that (1-5) and (1-6) hold. Passing, if necessary, to a telescoping, we may assume without loss of generality that $\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty$. For each $n > 0$, there is a unique $i_n \in \{1, \dots, r_n - 1\}$ such that

$$(A-1) \quad (i_n - 1)h_n + \sum_{j=r_n-i_n}^{r_n} \sigma_n(j) \leq \max_{1 \leq j < r_n} \sigma_n(j) < i_n h_n + \sum_{j=r_n-i_n+1}^{r_n} \sigma_n(j).$$

Since T is finite measure preserving, $\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{r_n} \sigma_n(i)}{r_1 \cdots r_n} < \infty$ and $\frac{h_n}{r_1 \cdots r_{n-1}} \rightarrow \mu(X) < \infty$ as $n \rightarrow \infty$. Therefore using the left-hand side inequality in (A-1) we obtain that $\sum_{n=1}^{\infty} \frac{(i_n-1)h_n}{r_1 \cdots r_n} < \infty$ and hence

$$(A-2) \quad \sum_{n=1}^{\infty} \frac{i_n}{r_n} < \infty.$$

Enumerate the elements of C_n in the increasing order: $C_n = \{c_n(0), \dots, c_n(r_n - 1)\}$ with $0 = c_n(0) < c_n(1) < \dots < c_n(r_n - 1)$. We now set

$$C_n^* := \{c_n(0), \dots, c_n(r_n - i_n - 1)\} \quad \text{and} \quad F_{n-1}^* := F_{n-1}$$

for each $n > 0$. Then the sequence $(C_n^*, F_{n-1}^*)_{n \geq 1}$ satisfies (I)–(III). Denote by $T^* = (T_i^*)_{i \in \mathbb{Z}}$ the associated (C, F) -action. Let X^* stand for the corresponding (C, F) -space of T^* . It is a routine to verify that $T^* \sim (r_n - i_n, \sigma_n^*)_{n=1}^\infty$, where

$$(A-3) \quad \sigma_n^*(i) := \begin{cases} \sigma_n(i), & \text{if } 1 \leq i < r_n - i_n \\ i_n h_n + \sum_{j=r_n-i_n+1}^{r_n} \sigma_n(j), & \text{if } j = r_n - i_n. \end{cases}$$

It follows from the right-hand side inequality in (A-1), (A-3) and Proposition A.2 that T is essentially 0-expansive. It remains to show that T^* is isomorphic to T . We do this in the same way as in the proof of [Da6, Theorem 2.8].

Given $n \geq 0$, since the restriction of μ to $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ is proportional to the infinite product of the equidistributions on F_n and C_j for all $j > n$ and (A-2) holds, it follows from the Borel-Cantelli lemma that the subset

$$Y_n := \{x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \mid c_j \in C_j^* \text{ eventually in } j\}$$

is of full measure in X_n . We now define a Borel map $\phi_n : Y_n \rightarrow X^*$ by setting

$$\phi_n(x) := (f_n + c_{n+1} + \cdots + c_m, c_{m+1}, c_{m+2}, \dots) \in F_m^* \times C_{m+1}^* \times C_{m+2}^* \times \cdots \subset X^*,$$

where m is chosen in such a way that $c_j \in C_j^*$ for all $j > m$. Then we have that $Y_n \subset Y_{n+1} \subset \cdots$. Hence the subset $Y := \bigcup_{n \geq 0} Y_n$ is of full measure in X . Moreover, $\phi_{n+1} \upharpoonright Y_n = \phi_n$ for each n . Hence we obtain a well-defined map $\phi : Y \rightarrow X^*$ by setting $\phi \upharpoonright Y_n = \phi_n$ for each $n > 0$. It is straightforward to verify that ϕ is a (measure theoretical) isomorphism of X and X^* . Moreover, ϕ intertwines T with T^* , as desired. \square

We now illustrate the explicit constructive procedure described in the proof of Theorem A.3 on the example of 2-adic odometer.

Example A.4. Let $T = (T_i)_{i \in \mathbb{Z}}$ be the 2-adic odometer, i.e. $T_1 \sim (r_n, \sigma_n)_{n=1}^\infty$, where $r_n = 2$ and $\sigma_n \equiv 0$ for all $n > 0$. Equivalently, T is the (C, F) -action associated with a sequence $(C_n, F_{n-1})_{n \geq 1}$ such that $F_n := \{0, 1, \dots, 2^n - 1\}$ and $C_{n+1} = \{0, 2^n\}$ for each $n \geq 0$. Of course, T is not essentially 0-expansive because $\phi^{(0)} \equiv 0$. Arguing as in the proof of Theorem A.3 and passing to a telescoping we first obtain that T is isomorphic to a rank-one action T' such that $T' \sim (2^n, \sigma'_n)$, where $\sigma'_n \equiv 0$ for all n . Then on every step of the cutting-and-stacking construction of T' we “replace” the last (i.e. 2^n -th) subtower¹⁹ with spacers. In such a way we obtain a new rank-one \mathbb{Z} -action $T^* \sim (2^n - 1, \sigma_n^*)_{n=1}^\infty$ such that

$$\sigma_n^*(i) = \begin{cases} 0, & \text{if } i = 1, \dots, 2^n - 2 \\ 2^{n(n+1)/2}, & \text{if } i = 2^n - 1 \end{cases}$$

for each n . By Proposition A.2, T^* is essentially 0-expansive. It is called a *constructive odometer* in [AdFePe]. By the proof of Theorem A.3, T^* is isomorphic to T .

¹⁹The height of this subtower is $2^{n(n+1)/2}$.

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